An Exposition of the Arzèla-Ascoli Theorem

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1 Introduction

The Arzèla-Ascoli theorem is a powerful result that provides sufficient conditions for the compactness of subsets of the space of continuous functions. It has a range of applications, from proving Peano's theorem in ordinary differential equations, to proving the Riemann mapping theorem in complex analysis. There are many methods to verify compactness, like checking for the general conditions of completenes and total boundedness. The Arzèla-Ascoli theorem allows one to verify more straightforward conditions, i.e. uniform boundedness and equicontinuity. In essence, it is a generalization of the Heine-Borel theorem in Euclidean space, which also provides "weaker" sufficient conditions for compactness of subsets. We begin with fundamental definitions and results from *Foundations of Mathematical Analysis* [1].

Metric spaces are useful mathematical objects that prescribe to sets the notion of distance. We define a metric and metric space as follows:

Definition 1. *Let M be a set. A **metric** on M is a function $d : M \times M \to [0, \infty)$ which satisfies*

(i) $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x, y) = d(y, x)$ for all $x, y \in M$

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$.

Definition 2. *A **metric space** is an ordered pair (M, d), where M is a set and d is a metric for M.*

Example 1. *The example that we will discuss in detail in this exposition is the space of continuous real valued functions on a compact interval. Let $I = [0, 1]$ and define $\mathcal{C}(I) = \{f : f$ is a continuous real-valued function on $I\}$. Then $(\mathcal{C}(I), d)$ is a metric space, when endowed with the supremum metric d, given by*

$$d(f, g) = ||f - g|| = \sup\{|f(x) - g(x)| : x \in I\}. \tag{1}$$

To define compactness of a metric space, we first introduce open sets and open covers.

Definition 3. *Let (M, d) be a metric space. Let $\epsilon > 0$ and let $x \in M$. We let*

$$B_\epsilon(x) = \{y \in M : d(x, y) < \epsilon\}. \tag{2}$$

*$B_\epsilon(x)$ is called the **open ball** of radius ϵ centered at x.*

Definition 4. *Let M be a metric space and let X be a subset of M. We say that X is **open** if for every $x \in X$, there exists an open ball $B_\epsilon(x)$, centered at x, such that $B_\epsilon(x) \subset X$.*

Definition 5. *An **open cover** of a metric space M is a collection $\mathcal{U}$ of open subsets of M such that $M = \bigcup \mathcal{U}$. A **subcover** of $\mathcal{U}$ is a subcollection $\mathcal{U}^*$ of $\mathcal{U}$ such that $M = \bigcup \mathcal{U}^*$.*

Definition 6. *Let M be a metric space, and let X be a subset of M. We say that a point $x \in M$ is a **limit point** of X if there is a sequence $\{x_n\}$ such that $x_n \in X$ for every positive integer n and $\lim_{n\to\infty} x_n = x$.*

Definition 7. *Let M be a metric sapce, and let X be a subset of M. We let $\bar{X}$ denote the set of limit points of X, called the **closure** of X.*

Definition 8. *A metric space M is **compact** if every open cover of M has a finite subcover. That is, let I be an index set and let $(U_\alpha)_{\alpha\in I}$ be a collection of open subsets of M such that $M = \bigcup_{\alpha\in I} U_\alpha$. Then M is compact if there exists a finite subset $J \subset I$ whereby*

$$M = \bigcup_{\alpha\in J} U_\alpha. \tag{3}$$

Definition 9. *Let (M, d) be a metric space. A subset $X \subset M$ is compact if and only if every open cover of X has a finite subcover.*

The Bolzano-Weierstrass characterization of sequential compactness is also particularly useful and will be used when we provide counterexamples to the Arzèla-Ascoli theorem.

Definition 10. *A metric space M is called **sequentially compact** if every sequence in M has a convergent subsequence.*

Theorem 1. *Let M be a metric space. Then M is compact if and only if M is sequentially compact.*

Proof. This is the Bolzano-Weirstrass characterization of a compact metric space. The proof of this theorem can be found on Page 150, Section 43 of [1]. □

There are many sufficient and necessary conditions for compactness. In this exposition, we will emphasize the ideas of completeness and total boundedness. We, thus, define the notion of completeness.

Definition 11. *Let M be a metric space. If every Cauchy sequence in M is convergent, we say that M is a **complete** metric space.*

Theorem 2. *The metric space $\mathbb{R}^n$ is complete.*

Proof. This theorem is proved on Page 150, Section 46 of [1]. □

Theorem 3. *The space of continuous real-valued functions on I, $\mathcal{C}(I)$, is a complete metric space.*

Proof. Fix $x_0 \in I$. Let $\{\phi_n\} \in \mathcal{C}(I)$ be a Cauchy sequence. Thus, there exists $N > 0$ such that for all $n, m > N$,

$$||\phi_n - \phi_m|| = \sup_{x\in I} |\phi_n(x) - \phi_m(x)| < \epsilon. \tag{4}$$

When $x = x_0$, there exists $N > 0$ such that

$$||\phi_n(x_0) - \phi_m(x_0)|| = \sup_{x\in I} |\phi_n(x) - \phi_m(x)| < \epsilon \tag{5}$$

for all $n, m > N$. Here, $\{\phi_n(x_0)\} \subset \mathbb{R}$ is a Cauchy sequence. Since $\mathbb{R}$ is a complete metric space by the aforementioned theorem, it follows that $\{\phi_n(x_0)\}$ is convergent. Since x_0 is arbitrary, this holds for all Cauchy sequences $\{\phi_n\} \in \mathcal{C}(I)$. Thus, $\mathcal{C}(I)$ is complete. □

Theorem 4. *A closet subset X of a complete metric space (M, d) is itself complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in X. Then, $\{x_n\}$ is Cauchy in (M, d). Hence, since M is complete, $\{x_n\} \to x \in M$ is convergent. Now, recall X is closed and hence contains all of its limit points; thus, $x \in X$. It follows that (X, d) is complete. □

The notion of total boundedness is an intuitive one: we can cover a metric space by finitely-many open balls.

Definition 12. *We say that a metric space M is **totally bounded** if for every $\epsilon > 0$, there exists $x_1, \ldots, x_n \in M$ such that $M = B_\epsilon(x_1) \cup \cdots \cup B_\epsilon(x_n)$.*

Theorem 5. *A metric space (M, d) is compact if and only if it is both complete and totally bounded.*

Proof. See Theorem 5.9 on Page 40 of [2]. □

2 Outline

The Arzèla-Ascoli theorem provides conditions for the compactness of subsets $\Phi \subset \mathcal{C}(I)$ of the space of continuous functions

$$\mathcal{C}(I) = \{f : f \text{ is a continuous real-valued function on } I\}.$$

That is, if Φ is uniformly bounded and equicontinuous, then its closure is compact. It is more efficient to verify the uniformly boundedness and equicontinuity compared to the general conditions of completeness and total boundedness. Thus, the Arzèla-Ascoli theorem is analogous to the Heine-Borel theorem for Euclidean space $\mathbb{R}^k$, in the sense that it reduces compactness to simpler conditions in a restricted context. In particular, the Heine-Borel theorem reduces compactness as follows:

Theorem 6 (Heine-Borel [1])**.** *Let S be a subset of $\mathbb{R}^k$, then S is compact if and only if S is closed and bounded.*

The strategy for proving the Arzèla-Ascoli theorem is as follows:

- To show that $\Phi \subset \mathcal{C}(I)$ is complete, we must show that it is complete and totally bounded.
- Since $\Phi \subset \mathcal{C}(I)$ is complete, it suffices to show total boundedness.
- We show that $\Phi \subset \mathcal{C}(I)$ can be approximated well by a finite set of piecewise linear functions $\psi_i \in \mathcal{C}(I)$, for $i = 1, \ldots, N$.
- Thus, we conclude that for all $\epsilon > 0$, there exists $\psi_i \in \mathcal{C}(I)$ such that $\Phi \subset B_\epsilon(\psi_1) \cup \cdots \cup B_\epsilon(\psi_N)$.

3 Theorem Statement & Proof

Theorem 7 (Arzèla-Ascoli)**.** *Let Φ be a subset of the space, $\mathcal{C}(I)$, of continuous real-valued functions on $I = [0, 1]$, equipped with the sup metric. Suppose that:*

*(a) There is some $B > 0$ such that $||\phi(x)|| \leq B$ for all $x \in I$ and all $\phi \in \Phi$. That is, Φ is **uniformly bounded**.*

*(b) For every $\epsilon > 0$ there is a $\delta > 0$ such that $||\phi(x) - \phi(y)|| < \epsilon$ for all $\phi \in \Phi$ whenever $|x - y| < \delta$, and $x, y \in I$. That is, Φ is **equicontinuous**.*

Then the closure $\bar{\Phi}$ is compact.

Proof. We must show that $\bar{\Phi}$ is compact. Recall $\mathcal{C}(I)$ is complete by Theorem 3. Invoking Theorem 4, since $\bar{\Phi}$ is closed and a subset of a complete metric space, it is also complete. Thus, by Theorem 5, given that $\bar{\Phi}$ is complete, it suffices to show that it is totally bounded.

Indeed, Φ being totally bounded is a necessary condition for $\bar{\Phi}$ to be totally bounded. Suppose Φ is totally bounded, then for every $\epsilon > 0$ there exists $\phi_1, \ldots, \phi_n \in \Phi \subset \bar{\Phi}$ such that

$$\Phi \subset \bigcup_{i=1}^{n} B_{\epsilon/2}(\phi_i). \tag{6}$$

Hence,

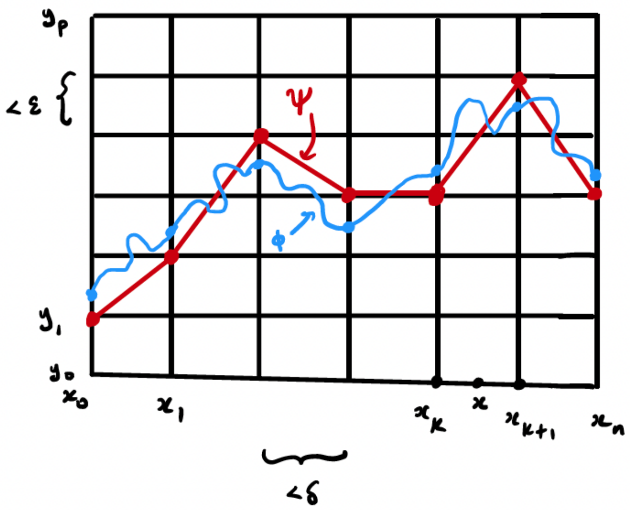
$$\bar{\Phi} \subset \overline{\bigcup_{i=1}^{n} B_{\epsilon/2}(\phi_i)} = \bigcup_{i=1}^{n} \overline{B_{\epsilon/2}(\phi_i)} \subset \bigcup_{i=1}^{n} B_{\epsilon}(\phi_i), \tag{7}$$

which means $\bar{\Phi}$ is totally bounded.

Hence, it suffices to show that Φ is totally bounded. Suppose $\epsilon > 0$ is given. We must cover Φ by finitely many balls of radius ϵ. Let B as in hypothesis (a) of the theorem and let $\delta > 0$ correspond to the given ϵ, as in hypothesis (b).

Divide I into subintervals, each of length less than δ, using finitely many points of subdivision: $0 = x_0 < x_1 < \cdots < x_n = 1$. Also divide $[-B, B]$ into subintervals of length less than ϵ using finitely many points of subdivision: $-B = y_0 < y_1 < \cdots < y_p = B$. Then the rectangle $I \times [-B, B]$ is divided into np smaller rectangles, each with width less than δ and height less than ϵ. This is shown in Figure 1.

Figure 1: Interpolation of continuous function $\phi \in \Phi$ with piecewise linear function $\psi \in \mathcal{C}(I)$ on interval I.



It is possible to associate to each $\phi \in \Phi$ a continuous, piecewise linear function, $y = \psi(x)$ whose graph has vertices all of which are of the form (x_k, y_l) for some $k \in \{1, \ldots, n\}, l \in \{1, \ldots, p\}$ and such that

$$||\psi(x_k) - \phi(x_k)|| < \epsilon \tag{8}$$

for all $k \in \{1, \ldots, n\}$. Such a ψ is shown in Figure 1, whereby we define $\psi(x_k)$ as the closest vertex to $\phi(x_k)$; this is clearly less than ϵ by construction of the grid above.

Consider $x \in I$. By construction of the grid, it must lie in an interval $x_k \leq x \leq x_{k+1}$, as shown in Figure 1. Hence consider:

$$\begin{aligned} ||\phi(x) - \psi(x)|| &= ||(\phi(x) - \phi(x_k)) + (\phi(x_k) - \psi(x_k)) + (\psi(x_k) - \psi(x))|| \\ &\leq ||\phi(x) - \phi(x_k)|| + ||\phi(x_k) - \psi(x_k)|| + ||\psi(x_k) - \psi(x)||. \end{aligned} \tag{9}$$

Since $|x - x_k| < \delta$, by equicontinuity of $\phi \in \Phi$, it follows that $||\phi(x) - \phi(x_k)|| < \epsilon$. Similarly, by construction of ψ, we have that $||\phi(x_k) - \psi(x_k)|| < \epsilon$. Hence,

$$||\phi(x) - \psi(x)|| < 2\epsilon + ||\psi(x_k) - \psi(x)|| \tag{10}$$

Finally, we bound the third term as follows:

$$\begin{aligned} ||\psi(x_k) - \psi(x)|| &= ||(\psi(x_k) - \phi(x_k)) + (\phi(x_k) - \phi(x_{k+1})) + (\phi(x_{k+1}) - \psi(x_{k+1}))|| \\ &\leq ||\psi(x_k) - \phi(x_k)|| + ||\phi(x_k) - \phi(x_{k+1})|| + ||\phi(x_{k+1}) - \psi(x_{k+1})||. \end{aligned} \tag{11}$$

Here, $||\psi(x_k) - \phi(x_k)|| < \epsilon$ and $||\phi(x_{k+1}) - \psi(x_{k+1})|| < \epsilon$ by construction of ψ. Likewise, $|x_k - x_{k+1}| < \delta$ and since $\phi \in \Phi$ is equicontinuous by hypothesis (b), it follows that $||\phi(x_k) - \phi(x_{k+1})|| < \epsilon$. Hence, the inequality in Equation (11) reduces to

$$||\psi(x_k) - \psi(x)|| < 3\epsilon. \tag{12}$$

Thus, Equation (10) becomes:

$$\begin{aligned} ||\phi(x) - \psi(x)|| &< 2\epsilon + ||\psi(x_k) - \psi(x)|| \\ &< 5\epsilon \end{aligned} \tag{13}$$

It follows that Φ can be covered by balls of radius 5ϵ whose centers are such functions ψ. We must show that there exist only finitely many such functions ψ. Recall, to any $\phi \in \Phi$ inside a rectangle, we associated a piecewise linear function that passes through vertices (x_k, y_k) for $k = 1, \ldots, n$. However, there are only finitely such functions; namely, for each x_k, there are p gridpoint choices for y_k for $k = 1, \ldots, n$. Hence, in total, there are only p^n such piecewise linear functions over interval I.

Indeed, this implies Φ can be covered by finitely many balls of radius 5ϵ with centers $\psi_i(x)$ for $i = 1, \ldots, N = p^n$. Thus,

$$\Phi \subset B_{5\epsilon}(\psi_1(x)) \cup \cdots \cup B_{5\epsilon}(\psi_N(x)), \tag{14}$$

which means that Φ is totally bounded, and the result follows at once. □

4 Counterexamples

We now enumerate two counterexamples where either hypotheses (a) or (b) of the theorem fail.

4.1 Example A

Let $I = [0, 1]$. Consider a subset $\Phi \subset \mathcal{C}(I)$ where hypothesis (a) fails, i.e. Φ is not uniformly bounded. Let

$$\Phi = \{\phi_n(x) = n : n = 1, \ldots, x \in I\} \subset \mathcal{C}(I, d_{\sup}). \tag{15}$$

We have that

$$||\phi_n(x)|| = \sup_{x \in I} |\phi_n(x)| = n \tag{16}$$

for all n. Thus, $|\phi_n(x)| \to \infty$ as $n \to \infty$ so Φ is not uniformly bounded.

We now check for compactness. Recall that $\bar{\Phi}$ is compact if Φ is compact. Moreover, by Theorem 1, Φ is compact if and only if it is sequentially compact. We say Φ is sequentially compact if every sequence has convergent subsequence in Φ. We will show not such convergent subsequence exists.

Proof. Let $\{n_k\}_{k\geq 1}$ be a subsequence. Suppose, for the sake of contradiction, that $\{\phi_{n_k}\}$ is a convergent subsequence, i.e. $\{\phi_{n_k}(x)\} \to \{\phi(x)\}$ for all $x \in I$. Then, for all $\epsilon > 0$, there exists $K > 0$ such that if $k \geq K$, then

$$||\phi_{n_k}(x) - \phi(x)|| < \epsilon \tag{17}$$

for all $x \in I$. Noting that Φ is not uniformly bounded:

$$\begin{aligned} \varepsilon &> ||\phi_{n_k}(x) - \phi(x)|| \\ &= \sup_{x\in I} |n_k - \phi(x)| \\ &\geq \left| n_k - \sup_{x\in I} \phi(x) \right| \\ &\geq n_k. \end{aligned} \tag{18}$$

Taking the limit as $k \to infty$, we get a contradiction, as $\{n_k\}$ is a subsequence so it is strictly increasing and positive. Hence, there does not exists a K such that if $k \geq K$ then $\varepsilon > n_k$. We have a contradiction, so Φ must not be sequentially compact. Hence, $\bar{\Phi}$ is not sequentially compact and, by Theorem 1, not compact. □

4.2 Example B

Let $I = [0, 1]$. Consider a subset $\Phi \subset \mathcal{C}(I)$ where hypothesis (b) fails, i.e. Φ is not equicontinuous. Let

$$\Phi = \{\phi_n(x) = x^n : n = 1, \ldots, x \in I\} \subset \mathcal{C}(I, d_{\text{sup}}). \tag{19}$$

We check that this is, indeed, not equicontinuous. Consider $0 < \delta < 1$. Moreover, suppose $x = 1, y = 1 - \delta/2$ then $|x - y| < \delta/2 < 1/2$. Hence,

$$|x^n - y^n| = 1 - \left(1 - \frac{\delta}{2}\right)^n. \tag{20}$$

Choose n such that $(1 - \delta/2)^n < 1/2$. Then $|x^n - y^n| > 1 - 1/2 = 1/2$, which means Φ is not equicontinuous.

We now prove that Φ is not sequentially compact and, hence, not compact.

Proof. Let $\{n_k\}_{k\geq 1}$ be a subsequence. As before, we proceed by contradiction. For suppose that $\{\phi_{n_k}\}$ is a convergent subsequence, i.e. $\{\phi_{n_k}(x)\} \to \{\phi(x)\}$ for all $x \in I$. Then, for all $\epsilon > 0$, there exists $K > 0$ such that if $k \geq K$, then

$$||\phi_{n_k}(x) - \phi(x)|| < \epsilon \tag{21}$$

for all $x \in I$. Hence, we choose another point $y \in I$ such that $|x - y| < \delta$ in the sense of equicontinuity of Φ. Now, consider:

$$\begin{aligned} ||\phi_{n_k}(x) - \phi(x)|| &= \sup_{x\in I} |x^{n_k} - \phi(x)| \\ &= \sup_{x\in I} |(x^{n_k} - y^{n_k}) + (y^{n_k} - \phi(x))| \\ &= \sup_{x\in I} |x^{n_k} - y^{n_k}| + \sup_{x\in I} |y^{n_k} - \phi(x)|, \end{aligned} \tag{22}$$

where we choose y such that equality holds in the third line. Since Φ is not equicontinuous, for all $\epsilon > 0$, there exists $\delta > 0$ such that $||\phi(x) - \phi(y)|| \geq \epsilon$ for all $\phi \in \Phi$ whenever $|x - y| < \delta$ and $x, y \in I$. Hence,

$$\begin{aligned} ||\phi_{n_k}(x) - \phi(x)|| &= \sup_{x \in I} |x^{n_k} - y^{n_k}| + \sup_{x \in I} |y^{n_k} - \phi(x)| \\ &\geq \epsilon + \sup_{x \in I} |y^{n_k} - \phi(x)| \\ &\geq \epsilon + \sup_{x \in I} |y^{n_k}| \\ &\geq \epsilon + |y^{n_k}|. \end{aligned} \tag{23}$$

Since $y \in I$, taking the limit as $k \to \infty$,

$$\lim_{k \to \infty} ||\phi_{n_k}(x) - \phi(x)|| \geq \epsilon, \tag{24}$$

a contradiction. So Φ is not sequentially compact. Hence, $\bar{\Phi}$ is not sequentially compact and, by Theorem 1, not compact.

□

References

[1] W.E. Pfaffenberger Richard Johnsonbaughm. *Foundations of Mathematical Analysis*. Dover, 2002.

[2] D.R. Wilkins. Course 212: Complete and compact metric spaces. 1991.