On the Time Evolution of Ricci Scalar Curvature in the Late Epoch for a ΛCDM-Parameterized Universe

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Sergio Charles

Contents

Abstract

We compute the evolution of the Ricci scalar curvature of a Lambda-Cold Dark Matter universe in the late epoch. The Friedmann–Lemaitre–Robertson–Walker metric, from which the Friedmann equations are derived, assumes a foliation of spacetime M as $\mathbb{R} \times \Sigma$ where Σ is a spatial three-manifold. We hence determine the equations of motion for non-minimally coupled inflation of a universe $M = \mathbb{R} \times \Sigma$ with a spatial three-manifold Σ of constant positive curvature whose density parameter is dominated by dark energy Λ.

1 Introduction

The large-scale dynamical evolution of the universe is governed by the Einstein field equations, which capture an equivalence between the curvature of spacetime and the presence of matter and energy. It is postulated that the accelerated expansion of the universe is driven by a negative pressure due to the existence of a positive dark, vacuum energy $\Lambda > 0$ permeating spacetime, as experimentally observed by Hubble in 1929 [11]. Under certain a priori assumptions and initial conditions, we may derive the evolution of the universe by characterizing geometric properties of spacetime such as curvature. As such, we seek to answer the following question.

Question. *What is the time evolution of Ricci scalar curvature in a ΛCDM universe during the late epoch?*

The analysis proceeds by identifying the large-scale dynamical evolution of such a universe via the non-vacuum Einstein field equations. By invoking the assumptions of a minimal six-parameter ΛCDM model, namely that spatial curvature vanishes and radiation density parameter contributions are negligible, we readily calculate the rate at which the universe is expanding, and thus determine an analytic solution to the Friedmann equations. By employing the Friedmann–Lemaitre–Robertson–Walker metric, we determine the Ricci scalar curvature of the universe and compute its asymptotic limit in the late epoch.

2 Pseudo-Riemannian Geometry and General Relativity

Spacetime $(M, \mathcal{A}, \nabla, g)$ is a four-dimensional Hausdorff differentiable topological manifold with a smooth atlas $\mathcal{A}$ carrying a torsion-free connection ∇ compatible with a Lorentzian metric g. An individual point in spacetime is henceforth referred to as an *event*. The path of a particle $\gamma : \mathbb{R}_{\geq 0} \to M$ parameterized by a one-dimensional set of elements is called a *worldline* [7]. Particles under the influence of no force have a worldline that is straight in spacetime. Any event in a flat spacetime, within the light cone, may be represented by a four-vector $X^\mu = (ct, x, y, z) = (x^0, x^1, x^2, x^4) \in \mathbb{R}^{1,3}$ for $\mu = 1, 2, 3, 4$, c the speed of light, and $\mathbb{R}^{1,3}$ the model Minkowski space. We can simply write the four-vector as $X^\mu = (ct, \boldsymbol{x})$ for $\boldsymbol{x} \in \mathbb{R}^3$.

Definition 2.1. *Let $(M, \mathcal{O}, \mathcal{A})$ be a smooth manifold and let $\gamma : \mathbb{R} \to M$ be a curve that is at least C^1, i.e. once continuously differentiable. Suppose $\gamma(\lambda_0) = p$. Then the velocity of p is the linear map $\nu_{\gamma,p} : C^\infty(M) \xrightarrow{\sim} \mathbb{R}$ where $\mathbb{R}$ has a vector space structure. The set $C^\infty(M)$ together with the two operations is a vector space of smooth functions $(C^\infty(M), \oplus, \otimes)$. The velocity $\nu_{\gamma,p} : C^\infty(M) \xrightarrow{\sim} \mathbb{R}$ is defined as $\nu_{\gamma,p}(f) := (f \circ \gamma)'|_{t=\lambda_0} = (f \circ \gamma)'(\lambda_0)$, evaluating f along γ and taking the derivative at λ_0. Likewise, the velocity acts as $f \mapsto \nu_{\gamma,p}(f) = (f \circ \gamma)'(\lambda_0)$. For each point $p \in M$ we define the tangent space to M at p to be:*

$$T_pM := \{\nu_{\gamma,p} : C^\infty(M) \xrightarrow{\sim} \mathbb{R} | \gamma \text{ smooth curve}\},$$

the set of all possible tangent vectors to all possible smooth curves through the point p. See Appendix A for a construction of the atlas on the tangent space of M.

Definition 2.2. *A Newtonian spacetime is a quintuple of structures $(M, \mathcal{O}, \mathcal{A}, \nabla, t)$ where $t : M \to \mathbb{R}$ is smooth absolute time t on M and $(M, \mathcal{O}, \mathcal{A})$ is a four-dimensional smooth manifold with topology $\mathcal{O}$ and atlas $\mathcal{A}$ given by the local chart $\phi : \mathcal{U} \to \mathcal{V}$, $\mathcal{U} \subset M$ and $V \subset \mathbb{R}^4$. The smooth time function satisfies $(dt)_p \neq 0$ for every $p \in M$ such that M is an absolute space (see Appendix A).*

We present Newton's laws of motion for a Newtonian spacetime which is curved.

(i) A body on which no force acts moves uniformly along a straight line, satisfying the autoparallel transport equation $\nabla_{\nu_\gamma}\nu_\gamma = 0$ for $\nabla_X Y$ the covariant derivative of the vector field Y with respect to the vector field X.

(ii) The deviation of a body's motion from a uniform motion is affected by a force reduced by a factor of the body's reciprocal mass. If there is no force, this reduces to the first law.

Gravity cannot be encoded in a curvature of space such that its effects show if particles under the influence of no other force move along straight lines in curved space. Let $F^\alpha = mf^\alpha$ be a gravitational force field, then

$$m\ddot{x}^\alpha(t) = F^\alpha(x(t)) \tag{1}$$

for $\alpha = 1, 2, 3$, where F^α satisfies the Poisson equation

$$-\partial_\alpha f^\alpha = 4\pi G\rho \tag{2}$$

for ρ the density of matter in the universe. Since the gravitational force is proportional to the mass of the body on which it acts, $m\ddot{x}^\alpha(t) = F^\alpha(x(t)) = mf^\alpha(x(t))$ so $\ddot{x}^\alpha(t) = f^\alpha(x(t))$ and the motion becomes mass independent, a property of the weak equivalence principle. We interpret $\ddot{x}^\alpha(t) = f^\alpha(x(t))$ as the autoparallel equation (see Appendix B):

$$\ddot{\gamma}^m_{(x)}(\lambda) + \Gamma^m_{(x)ab}(\gamma(\lambda))\dot{\gamma}^a_{(x)}(\lambda)\dot{\gamma}^b_{(x)}(\lambda) = 0 \tag{3}$$

for $\gamma^m_{(x)} = x^m \circ \gamma$ and $\Gamma^m_{(x)ab}$ connection coefficients of the connection ∇ on $(M, \mathcal{O}, \mathcal{A})$. However, one cannot find connection coefficients such that Newton's equation takes the form of an autoparallel equation. Since a particle may be parameterized in worldline coordinates by $X = (X^0, X^1, X^2, X^3) = (ct, x^1(t), x^2(t), x^3(t))$, $\dot{X}^0 = 1$ so $\ddot{X}^0 = 0$. Observe that $\ddot{X}^\alpha$ for $\alpha = 1, 2, 3$ is the same as $\ddot{x}^\alpha$ for $\alpha = 1, 2, 3$. We may set the speed of light c to unity by convention. Thus, $x^\alpha = X^\alpha$ for $\alpha = 1, 2, 3$ so $\ddot{x}^\alpha - f^\alpha(x(t)) = 0$ or

$$\begin{aligned} &\ddot{X}^0 = 0, \\ &\ddot{X}^\alpha - f^\alpha(X(t)) = 0 \qquad (\alpha = 1, 2, 3). \end{aligned} \tag{4}$$

Since $\dot{X}^0 = 1$, the second equation becomes $\ddot{X}^\alpha - f^\alpha(X(t))\dot{X}^0\dot{X}^0 = 0$ for $\alpha = 1, 2, 3$. Thus,

$$\ddot{X}^a + \Gamma^a_{bc}\dot{X}^b\dot{X}^c \tag{5}$$

for $a = 0, 1, 2, 3$, which is the autoparallel equation in spacetime. We can impose the equivalence by choosing the connection coefficients Γ where $\ddot{X}^0 + \Gamma^0_{bc}\dot{X}^b\dot{X}^c = 0$ for $a = 0$ so $\Gamma^0_{bc} = 0$ corresponds to $\ddot{X} = 0$. To reconcile $\ddot{X}^\alpha + \Gamma^\alpha_{bc}\dot{X}^b\dot{X}^c = 0$ with $\ddot{X}^\alpha - f^\alpha(X(t))\dot{X}^0\dot{X}^0 = 0$, we let

$$\begin{aligned} &\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{0\beta} = \Gamma^\alpha_{\beta 0} = 0, \\ &\Gamma^\alpha_{00} = -f^\alpha. \end{aligned} \tag{6}$$

In the local chart $(\mathcal{U}, x)$, $\Gamma^{\alpha}_{00} = -f^{\alpha} = -\frac{1}{m}F^{\alpha}$ and $\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{0\beta} = \Gamma^{\alpha}_{\beta 0} = 0$. We can calculate the Riemann curvature for ∇ defined by the Γ^{i}_{jk} above. Namely, Riem $\left(dx^{\alpha}, \frac{\partial}{\partial X^0}, \frac{\partial}{\partial X^{\beta}}, \frac{\partial}{\partial X^0}\right) = R^{\alpha}_{0\beta 0} = -\partial_{\beta} f^{\alpha}$. The Ricci tensor is the contraction of the Riemann curvature tensor $R_{00} = R^{m}_{0m0} = -\partial_{\alpha} f^{\alpha}$ so $R_{00} = 4\pi G\rho$. Note, density ρ can be expressed as the $(0,0)$-component of the energy-momentum tensor: $T_{00} = \rho/2$. If Φ is the gravitational potential of a gravitational field $\boldsymbol{g}$ as a function of spacetime, then $\nabla\Phi = -\boldsymbol{g}$ and $-\nabla \cdot \boldsymbol{g} = 4\pi G\rho$ by Gauss's law. Furthermore,

$$\nabla^2 \Phi = 4\pi G\rho. \tag{7}$$

Particles under the influence of no force have worldlines that are straight in spacetime.

Definition 2.3. *A vector $X \in T_pM$ in spacetime is called*

(i) future-directed if $dt(X) > 0$ for $X \in \Gamma(TM)$.

(ii) spatial if $dt(X) = 0$.

(iii) past-directed if $dt(X) < 0$, where $dt : T_pM \xrightarrow{\sim} \mathbb{R}$.

The worldline of a particle under the influence of no force is a future-directed autoparallel with respect to ∇ on spacetime. If F is a spatial vector field then the equation of motion for a worldline in the presence of gravity F is (see Appendix E):

$$\ddot{X}^{\alpha} + \Gamma^{\alpha}_{\gamma\delta}\dot{X}^{\gamma}\dot{X}^{\delta} + \Gamma^{\alpha}_{00}\dot{X}^{0}\dot{X}^{0} + 2\Gamma^{\alpha}_{\gamma 0}\dot{X}^{\gamma}\dot{X}^{0} = \frac{F^{\alpha}}{ma^2} \tag{8}$$

where a depends on the unit measure of time used and Newtonian spacetime is torsion-free. The left-hand side represents the vector field components of acceleration and $\Gamma^{\alpha}_{00} = -f^{\alpha}$ in the presence of gravity. See Appendix A for a review of Minkowski spacetime.

Definition 2.4. *The four-velocity is the derivative of X^{μ} with respect to proper time τ because it does not transform under Lorentz transformation. Thus, if c is the speed of light and v is the magnitude of the three-velocity,* $\nu^{\mu} = \frac{dX^{\mu}}{d\tau} = \left(\frac{dX^0}{d\tau}, \frac{dX^1}{d\tau}, \frac{dX^2}{d\tau}, \frac{dX^3}{d\tau}\right) = \frac{dX^{\mu}}{dt}\frac{dt}{d\tau} = \frac{1}{\sqrt{1-v^2/c^2}}\frac{dX^{\mu}}{dt} = \left(\frac{c}{\sqrt{1-v^2/c^2}}, \frac{1}{\sqrt{1-v^2/c^2}}\frac{d\boldsymbol{x}}{dt}\right)$.

Likewise, we consider the trajectory $\gamma : \mathbb{R} \to M$ of a particle in the spacetime M. Then γ is a geodesic if and only if it satisfies the Euler–Lagrange equations for the Lagrangian

$$\begin{aligned} \mathcal{L} :& TM \to \mathbb{R} \\ & X \mapsto \sqrt{g(X,X)} \end{aligned} \tag{9}$$

where $g(X,X) : \Gamma(TM) \times \Gamma(TM) \xrightarrow{\sim} C^{\infty}(M)$. In a local chart $(\mathcal{U}, x)$ of spacetime with connection coefficients $\Gamma^{m}_{(x)\mu\nu}$, the Euler–Lagrange geodesic equations describing the trajectory γ of a particle are given by (see Appendix B):

$$\ddot{\gamma}^{\rho}_{(x)}(\lambda) + \Gamma^{\rho}_{(x)\mu\nu}(\gamma(\lambda))\dot{\gamma}^{\mu}\dot{\gamma}^{\nu}. \tag{10}$$

3 Deriving the Non-Vacuum Einstein Field Equations

Definition 3.1. *Consider a relativistic Lorentzian spacetime* $(M, \mathcal{O}, \mathcal{A}^{\uparrow}, \nabla, g, T)$ *with absolute time* t *orientation* T. *An observer is a worldline* γ *with* $g(\nu_\gamma, \nu_\gamma) > 0$ *and* $g(\gamma, \nu_\gamma) > 0$ *together with a choice of basis* $e_0(\lambda) = \nu_{\gamma,\gamma(\lambda)}, e_1(\lambda), e_2(\lambda), e_3(\lambda)$ *of each* $T_{\gamma(\lambda)}M$ *where the observer worldline passes if* $g(e_\mu(\lambda), e_\nu(\lambda)) = \eta_{\mu\nu}$. *An observer is therefore a smooth curve in the frame bundle* LM *over* M.

Here, the inner product on M is defined by the metric as:

$$g(e_\mu(\lambda), e_\nu(\lambda)) = \eta_{\mu\nu} := \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}_{\mu\nu} \tag{11}$$

so the basis vectors are orthonormal as a Lorentzian metric with signature $(+ - - -)$.

Definition 3.2. *A clock carried by a specific observer* $(\gamma, e) = (\gamma, e_0(\lambda), e_1(\lambda), e_2(\lambda), e_3(\lambda))$ *will measure a time, namely a proper time or eigentime, given by*

$$\tau := \int_{\lambda_0}^{\lambda_1} d\lambda \sqrt{g_{\gamma(\lambda)}(\nu_{\gamma,\gamma(\lambda)}, \nu_{\gamma,\gamma(\lambda)})} \tag{12}$$

between the events $\gamma(\lambda_0)$, *starting the clock, and* $\gamma(\lambda_1)$, *stopping the clock.*

The action of a massive, future-directed particle worldline of mass m is

$$S_{\text{massive}}[\gamma] = m \int d\lambda \sqrt{g_{\gamma(\lambda)}(\nu_{\gamma,\gamma(\lambda)}, \nu_{\gamma,\gamma(\lambda)})} \tag{13}$$

with the proper-time orientation $g_{\gamma(\lambda)}(\gamma_{\gamma(\lambda)}, \nu_{\gamma,\gamma(\lambda)}) > 0$. Similarly, the action of a massless particle worldline with Lagrange multiplier μ is

$$S_{\text{massless}}[\gamma, \mu] = \int d\lambda \mu g \left(\nu_{\gamma,\gamma(\lambda)}, \nu_{\gamma,\gamma(\lambda)}\right). \tag{14}$$

Composite systems with coupled fields have an action given by the sum of actions of the independent part in addition to a non-linear interaction term. Consider the action $S[\gamma]$ of one particle and an action $S[\delta]$ of an another coupled particle. The interaction term is $S_{\text{int}}[\gamma, \delta]$, so the composite action is $S[\gamma] + S[\delta] + S_{\text{int}}[\gamma, \delta]$, as seen in electromagnetic field theories due to Maxwell (see Appendix D).

We consider a field theory in which the local generalized coordinates are a set of fields $\Phi^i(x)$. The action S is an integral over the space of Lagrangian density $\mathcal{L}$ [2]:

$$S = \int d^n x \mathcal{L}(\Phi^i, \nabla_\mu \Phi^i) \tag{15}$$

where $\mathcal{L} = \sqrt{-g}\hat{\mathcal{L}}$ for $\hat{\mathcal{L}}$ a scalar. The corresponding Euler–Lagrange equations become:

$$\frac{\partial \hat{\mathcal{L}}}{\partial \Phi} - \nabla_\mu \left(\frac{\partial \hat{\mathcal{L}}}{\partial(\nabla_\mu \phi)} \right) = 0. \tag{16}$$

In fact, the action for a single scalar field ϕ is:

$$S[\phi] = \int_M d^n x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} (\nabla_\mu \phi)(\nabla_\nu \phi) - V(\phi) \right] \tag{17}$$

which has Euler–Lagrange equations of motion given by

$$\nabla^\mu \nabla_\mu \phi - \frac{dV}{d\phi} := \Box \phi - \frac{dV}{d\phi} = 0. \tag{18}$$

We henceforth set the speed of light to $c = 1$.

The Ricci scalar is the independent scalar obtained from the metric tensor with at most second-order derivatives. Therefore, the simplest choice for the Lagrangian is the Ricci scalar from which we obtain the so-called Einstein–Hilbert action:

$$S_{\text{Hilbert}}[g] = \int d^x \sqrt{-g} R \tag{19}$$

which is diffeomorphism invariant. We need to write down an action for the metric tensor field itself. So the action $S_{\text{grav}}[g]$ will be added to carry $S_{\text{matter}}[A, \phi, \dots]$ in order to describe the total system. Therefore,

$$S_{\text{total}}[g, A] = S_{\text{grav}}[g] + S_{\text{Maxwell}}[A, g]. \tag{20}$$

If we vary the total action with respect to g, then there is a contribution from both $S_{\text{grav}}[g]$ and $S_{\text{Maxwell}}[A, g]$. In fact, the variation of $S_{\text{grav}}[g]$ with respect to g is called G_{ab}, the *Einstein tensor*, and the variation of $S_{\text{Maxwell}}[A, g]$ with respect to g is called $-T_{ab}$, the *energy-momentum tensor*.

For any matter field Φ, $S_{\text{matter}}[\Phi, g]$ is a matter action and the energy-momentum tensor is the $(2,0)$-tensor

$$T^{\mu\nu} := -\frac{2}{\sqrt{-g}} \left(\frac{\partial \mathcal{L}_{\text{matter}}}{\partial g_{\mu\nu}} - \partial_\lambda \frac{\partial \mathcal{L}_{\text{matter}}}{\partial(\partial_\lambda g_{\mu\nu})} \right). \tag{21}$$

This originates from the contribution due to the variation of $S_{\text{matter}}[\Phi, g] = \int_M d^4 x \mathcal{L}_{\text{matter}}(g; \Phi)$ where $T : \Gamma(T^*M) \times \Gamma(T^*M) \xrightarrow{\sim} C^\infty(M)$. The variation of the Einstein–Hilbert action $S_{\text{Hilbert}}[g] = \int_M d^4 x \sqrt{-g} R_{\mu\nu} g^{\mu\nu}$ with respect to the metric is:

$$\begin{aligned} \delta S_{\text{Hilbert}}[g] &= \int_M d^4 x \delta(\sqrt{-g} R_{\mu\nu} g^{\mu\nu}) \\ &= \int_M d^4 x [(\delta \sqrt{-g}) R_{\mu\nu} g^{\mu\nu} + \sqrt{-g} \delta g^{\mu\nu} R_{\mu\nu} + \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu}]. \end{aligned} \tag{22}$$

Let $(\delta S)_1 = \int_M d^4 x R \delta \sqrt{-g}$, $(\delta S)_2 = \int_M d^4 x \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu}$, and $(\delta S)_3 = \int_M d^4 x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu}$. Observe, the Ricci tensor is the contraction of the Riemann curvature tensor

$$R^\rho_{\mu\lambda\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\mu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}. \tag{23}$$

The variation of the Riemann curvature tensor is thus

$$\begin{aligned} \delta R^\rho_{\sigma\mu\nu} &= \partial_\mu \delta \Gamma^\rho_{\nu\sigma} - \partial_\nu \delta \Gamma^\rho_{\mu\sigma} + \delta \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} \lambda - \delta \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} - \Gamma^\rho_{\nu\lambda} \delta \Gamma^\lambda_{\mu\sigma} \\ &= \nabla_\mu (\delta \Gamma^\rho_{\nu\sigma}) - \nabla_\nu (\delta \Gamma^\rho_{\mu\sigma}) \\ &:= (\delta \Gamma)^\rho_{\nu\sigma;\mu} - (\delta \Gamma)^\rho_{\mu\sigma;\nu}. \end{aligned} \tag{24}$$

By contracting two indices, we obtain the Palatini identity:

$$\delta R_{\sigma\nu} = \nabla_\rho (\delta \Gamma^\rho_{\nu\sigma}) - \nabla_\nu (\delta \Gamma^\rho_{\rho\sigma}). \tag{25}$$

Since $g^{\mu\nu}g_{\lambda\nu} = \delta^{\mu}_{\lambda}$, we may determine the variations of the metric and the inverse metric in terms of one another by

$$\delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}. \tag{26}$$

Considering $(\delta S)_1$, if $M \in \mathrm{Mat}_{n\times n}(K)$ has non-vanishing determinant over the field K, then $\ln(\det M) = \mathrm{Tr}(\ln M)$ where $\exp(\ln M) = M$. The variation of this identity is $\frac{1}{\det M}\delta(\det M) = \mathrm{Tr}(M^{-1}\delta M)$. Setting $M = g_{\mu\nu}$ and $\det M = \det g_{\mu\nu} := g$, we find

$$\delta g = g(g^{\mu\nu}\delta g_{\mu\nu}) = -g(g_{\mu\nu}\delta g^{\mu\nu}). \tag{27}$$

By the above,

$$\begin{aligned}\delta\sqrt{-g} &= -\frac{1}{2\sqrt{-g}}\delta g \\ &= \frac{g}{2\sqrt{-g}}g_{\mu\nu}\delta g^{\mu\nu} \\ &= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}.\end{aligned} \tag{28}$$

Let A and B be $(1,1)$-tensors, then $(\nabla_\nu A)^i_j = A^i_{j;\nu}$ and $(\nabla_\nu B)^i_j = B^i_{j;\nu}$. Assuming the metric is compatible, i.e. $\nabla g = 0$,

$$\begin{aligned}\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} &= \sqrt{-g}g^{\mu\nu}[(\delta\Gamma)^m_{\mu\lambda;\nu} - (\delta\Gamma)^\lambda_{\mu\nu;\lambda}] \\ &= \sqrt{-g}(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu})_{;\lambda}.\end{aligned} \tag{29}$$

Thus, $\nabla_\nu(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda}) = (\nabla_\nu g^{\mu\nu})\delta\Gamma^\lambda_{\mu\lambda} + g^{\mu\nu}(\nabla_\nu\delta\Gamma^\lambda_{\mu\lambda}) = g^{\mu\nu}\nabla_\nu\delta\Gamma^\lambda_{\mu\lambda}$. The expression thereby reduces to

$$\begin{aligned}\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} &= \sqrt{-g}(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda})_{;\nu} - \sqrt{-g}(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu})_{;\lambda} \\ &= (\sqrt{-g}A^\nu)_{;\nu} - (\sqrt{-g}B^\nu)_{;\nu}.\end{aligned} \tag{30}$$

Adding the respective terms for the different variations, we obtain the total variation

$$\delta S_{\text{Hilbert}} = \int_M d^4x\left[\frac{1}{2}\sqrt{-g}g^{\lambda\rho}\delta g_{\lambda\rho}g^{\mu\nu}R_{\mu\nu} - \sqrt{-g}g^{\mu\lambda}g^{\nu\rho}\delta g_{\lambda\rho}R_{\nu\nu} + (\sqrt{-g}A^\nu)_{;\nu} - (\sqrt{-g}B^\nu)_{;\nu}\right] \tag{31}$$

where the Gibbons–Hawking–York boundary divergence surface term $(\sqrt{-g}A^\nu)_{;\nu} - (\sqrt{-g}B^\nu)_{;\nu}$ vanishes at the boundary of M. Thus,

$$\delta S_{\text{Hilbert}} = \int_M d^4x\sqrt{-g}\delta g^{\mu\nu}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right] \tag{32}$$

for an arbitrary variation $g^{\mu\nu}$. Recall, the derivative of the action varied over a set of fields $\{\Phi^i\}$ satisfies [3]:

$$\delta S = \int_M \sum_i \left(\frac{\delta S}{\delta\Phi^i}\delta\Phi^i\right) d^n x. \tag{33}$$

It follows that the Einstein field equations in a vacuum are:

$$\begin{aligned}0 &= \frac{1}{\sqrt{-g}}\frac{\delta S_{\text{Hilbert}}}{\delta g^{\mu\nu}} \\ &= R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}.\end{aligned} \tag{34}$$

We would like to determine the non-vacuum Einstein field equations given by the Einstein–Hilbert action with an additional Lagrangian density $\mathcal{L}_M$ describing matter fields. That is, we consider

$$S = \frac{1}{16\pi G} S_{\text{Hilbert}} + S_M := \int_M d^4x \sqrt{-g} \left[\frac{1}{16\pi G} R + \mathcal{L}_M \right]. \tag{35}$$

By following a similar variational procedure, we find

$$\begin{aligned} 0 &= \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \\ &= \frac{1}{16\pi G} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}. \end{aligned} \tag{36}$$

Defining the energy-momentum tensor as

$$T_{\mu\nu} = -2 \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} \tag{37}$$

we recover the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{38}$$

If we modify the action by the so-called cosmological constant Λ to

$$S = \int_M d^4x \sqrt{-g} \left[\frac{1}{16\pi G} (R - 2\Lambda) + \mathcal{L}_M \right] \tag{39}$$

and refactor the speed of light c, we obtain the modified Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{40}$$

a set of ten independent equations. In 1915, Einstein set $\Lambda < 0$ in order to obtain a static, non-expanding universe. In 1929, Hubble discovered that the universe is expanding at an accelerated rate and set $\Lambda = 0$. Presently, it is known that $\Lambda > 0$, albeit very small. This term in the modified Einstein field equations is called dark energy; it does not interact with anything, but it contributes to the curvature of spacetime. There is dark energy everywhere in the spacetime M provided by the Λ. It is hypothesized that such energy is the vacuum fluctuations of primordial quantum fields in spacetime due to the uncertainty principle of quantum mechanics.

The definition of the energy-momentum tensor in Equation (37) also agrees with a scalar field theory. Consider the action for a scalar field ϕ with potential $V(\phi)$:

$$S[\phi] = \int_M d^nx \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} (\nabla_\mu \phi)(\nabla_\nu \phi) - V(\phi) \right], \tag{41}$$

and vary it with respect to the inverse metric as follows.

$$\begin{aligned} \delta S[\phi] &= \int_M d^nx \left[\sqrt{-g} \Big(-\frac{1}{2} \delta g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \Big) + \delta \sqrt{-g} \Big(-\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \Big) \right] \\ &= \int_M d^nx \sqrt{-g} \delta g^{\mu\nu} \left[-\frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi + \Big(-\frac{1}{2} g_{\mu\nu} \Big) \Big(-\frac{1}{2} g^{\rho\sigma} \nabla_\rho \phi \nabla_\sigma \phi - V(\phi) \Big) \right] \end{aligned} \tag{42}$$

Thus, applying the definition of the energy-momentum tensor, we have

$$\begin{aligned} T_{\mu\nu}[\phi] &= -2 \frac{1}{\sqrt{-g}} \frac{\delta S[\phi]}{\delta g^{\mu\nu}} \\ &= \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \nabla_\rho \phi \nabla_\sigma \phi - g_{\mu\nu} V(\phi), \end{aligned} \tag{43}$$

which is correct for a scalar field theory.

4 The Friedmann–Lemaitre–Robertson–Walker Metric and the Expanding Universe

The universe is postulated to be a priori spatially homogeneous and isotropic, such that it may be foliated into maximally symmetric space-like slices. Thus, the spacetime M may be written as $\mathbb{R} \times \Sigma$ where $\mathbb{R}$ represents the temporal dimension and Σ is a maximally symmetric three-manifold. The spacetime metric has a line component of the form

$$ds^2 = -dt^2 + R^2(t) d\mathbf{\Sigma}^2 \tag{44}$$

where $R(t)$ is the scale factor, a function of time, and $d\mathbf{\Sigma}^2$ is the metric on Σ which may be written as

$$d\mathbf{\Sigma}^2 = \gamma_{ij}(x) dx^i dx^j \tag{45}$$

for (x^1, x^2, x^3) comoving coordinates on Σ and γ_{ij} the maximally symmetric three-dimensional metric. Intuitively, the scale factor $R(t)$ is a measure of the radius of the time slice at a given time t. Maximally symmetric metrics satisfy

$$R_{ijkl} = k(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}) \tag{46}$$

for k the Gaussian curvature of Σ. We normalize the curvature k to either $+1, 0, -1$, corresponding to different maximally symmetric spatial universes. If $k = -1$, Σ has constant negative curvature and the universe is *open* or *hyperbolic*. If $k = 0$, Σ has no curvature and the universe is *flat* or *Euclidean*. If $k = +1$, Σ has constant positive curvature and the universe is *closed* or *parabolic*. Thus, in a homogeneous and isotropic universe, for τ proper time, the metric line element is

$$ds^2 = -c^2 d\tau^2 = -c^2 dt^2 + a(t)^2 d\mathbf{\Sigma}^2 \tag{47}$$

for $\mathbf{\Sigma}$ ranging over a maximally symmetric three-manifold of uniform Gaussian curvature, i.e. an elliptic, Euclidean, or hyperbolic space. The metric on a homogeneous and isotropic, maximally symmetric hyper-surface in polar coordinates (r, θ, ϕ) may be written as

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[\frac{dr}{1 - kr^2} + r^2 d\mathbf{\Omega}^2 \right] \tag{48}$$

where $d\mathbf{\Omega}^2 = d\theta^2 + \sin^2\theta d\phi^2$. When $|k|= 1$, the radius of curvature of M is simply $a(t)$. We compute the Christoffel symbols and, hence, Ricci curvature [3]:

$$\begin{aligned}
\Gamma^0_{11} &= \frac{a\dot{a}}{1 - kr^2}, & \Gamma^1_{11} &= \frac{kr}{1 - kr^2} \\
\Gamma^0_{22} &= a\dot{a}r^2, & \Gamma^0_{33} &= a\dot{a}r^2 \sin^2\theta \\
\Gamma^1_{01} &= \Gamma^2_{02}, & \Gamma^3_{03} &= \frac{\dot{a}}{a} \\
\Gamma^1_{22} &= -r(1 - kr^2), & \Gamma^1_{33} &= -r(1 - kr^2)\sin^2\theta \\
\Gamma^2_{12} &= \Gamma^3_{13}, & & \\
\Gamma^2_{33} &= -\sin\theta\cos\theta, & \Gamma^3_{23} &= \cot\theta.
\end{aligned} \tag{49}$$

As such, the non-zero components of the Ricci tensor are

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a} \\ R_{11} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1 - kr^2} \\ R_{22} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k) \\ R_{33} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k)\sin^2\theta, \end{aligned} \tag{50}$$

from which it follows that the Ricci scalar curvature is

$$R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} + \frac{k}{a^2}\right)^2\right]. \tag{51}$$

The expansion of the universe is uniquely parameterized by the dimensionless scale factor $a := a(t)$ for $0 \leq t < \infty$. At present day t_0, the scale factor is $a_0 := a(t_0)$, which is an integration constant determined by initial conditions.

5 Deriving the Ricci Scalar Curvature in a ΛCDM Model

The matter and energy in a spatially homogeneous and isotropic universe may be modelled by a perfect fluid. Moreover, the perfect fluid will be at rest in comoving coordinates. Thus, the four-velocity is the time-like vector field

$$U^\mu = (1, 0, 0, 0), \tag{52}$$

normalized such that $g(U, U) = 1$. The energy-momentum tensor is [3]:

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + pg^{\mu\nu} \tag{53}$$

or, equivalently,

$$\begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g^{ij}p & \\ 0 & & & \end{bmatrix}. \tag{54}$$

It assumes the form $T^\mu_\nu = \text{diag}(-\rho, p, p, p)$ with a trace of $T = T^\mu_\mu = -\rho + 3p$. The radiation fluid must satisfy $T^{\mu\nu}_{\text{p.f.}} g_{\mu\nu} = 0$ such that $\rho = 3p$ or

$$p = \frac{1}{3}\rho, \tag{55}$$

which is its equation of state. That is, the energy-momentum tensor is trace-free. If $p = 0$, we have dust which still has a density but does not develop pressure in the early universe. Consider the zero component of the energy conservation equation [3]:

$$0 = \nabla_\mu T^\mu_0 = \partial_\mu T^\mu_0 + \Gamma^\mu_{\mu\lambda} T^\lambda_0 - \Gamma^\lambda_{\mu 0} T^\mu_\lambda. \tag{56}$$

Definition 5.1. *A relation $p = \Phi(\rho)$ is called an equation of state for a perfect fluid of density ρ and pressure p.*

ω	Matter field
$1/3$	Radiation, relativistically momentous particles
0	Dust
-1	Cosmological constant (negative pressure $p = -\rho c^2$)
$-k/3$	Spatial curvature

Figure 1: Matter fields for different equations of state ω.

Perfect fluids behave according to the equation of state

$$\omega = \frac{p}{\rho}, \tag{57}$$

whereby the parameter ω corresponds to different matter fields (see Figure 1). Thus, the conservation of energy becomes

$$\frac{\dot{\rho}}{\rho} = -3(1+\omega)\frac{\dot{a}}{a}, \tag{58}$$

which may then be integrated to obtain the relation $\rho \propto a^{-3(1+\omega)}$. The dominant energy condition requires that matter does not destabilize, which means $|\omega| \leq 1$. Dust is a matter type consisting of non-relativistic particles with approximately zero pressure $p_M = 0$. Such a universe is said to be *matter-dominated* and the energy density varies as $\rho_M \propto a^{-3}$. Radiation refers to either electromagnetic radiation or massive particles moving at relativistic speeds. A *radiation-dominated universe* is one in which the majority of energy density is derived from radiation. The energy density behaves asymptotically as $\rho_\gamma \propto a^{-4}$. Today, the ratio of matter density to radiation density is on the order of $\rho_M/\rho_\gamma \propto 10^3$. Vacuum energy, due to the curvature of the maximally symmetric hypersurface $\mathbb{R} \times \Sigma$, behaves like a perfect fluid with equation of state $p_\Lambda = -\rho_\Lambda$. The energy density is therefore a constant $\rho_\Lambda \propto a^0$. Energy density, in the form of matter and radiation, decreases exponentially as the universe expands so a nonzero vacuum energy dominates the universe at large times. The universe is then said to be *vacuum-dominated*, examples of which include de Sitter and anti-de Sitter space. Inserting the ansatz for $T^{\mu\nu}$ in the Einstein field equations

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right),$$

the $(0,0)$-component equation reads

$$-3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3p) \tag{59}$$

and the general (i,j)-component equation gives

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G(\rho - p). \tag{60}$$

We eliminate second-order derivatives in Equation (59) and, therefore, conclude:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{61}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{62}$$

Re-introducing the cosmological constant into these equations, we obtain:

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3p)a + \frac{\Lambda}{3},$$
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}, \tag{63}$$

which are known as the *Friedmann equations.* Note, the first Friedmann equation may be used to obtain the conservation law $T^{\mu\nu}_{;\nu} = 0$.

The rate of expansion of the universe is given by the *Hubble parameter*

$$H = \frac{\dot{a}}{a}. \tag{64}$$

The Hubble parameter in the present epoch is known as the *Hubble constant* $H_0 \sim 70{\pm}10 \text{kms}^{-1}\text{Mpc}^{-1}$.

Definition 5.2. *The density parameter is given by*

$$\Omega = \frac{8\pi G}{3H^2}\rho = \frac{\rho}{\rho_{crit}} \tag{65}$$

where the critical density

$$\rho_{crit} = \frac{3H^2}{8\pi G} \tag{66}$$

is the density needed for arrested expansion $a(t) = 0$ of the universe at infinite time, for which the spatial three-manifold Σ is Euclidean. A universe with critical density is said to be flat.

Thus, the first Friedmann equation may be written as

$$\Omega - 1 = \frac{k}{H^2 a^2}, \tag{67}$$

which corresponds to a universe of fixed density parameter and spatial curvature (see Figure 2).

Density ρ	Density parameter Ω	Spatial curvature k	Type of maximally symmetric three-manfiold Σ
$\rho < \rho_{\text{crit}}$	$\Omega < 1$	$k < 0$	open
$\rho = \rho_{\text{crit}}$	$\Omega = 1$	$k = 0$	flat
$\rho > \rho_{\text{crit}}$	$\Omega > 1$	$k > 0$	hyperbolic

Figure 2: Density ρ, density parameter Ω, and spatial curvature k for different types of maximally symmetric three-manifolds Σ.

Using empirical observation, it is believed $\Omega \sim 1$. By integrating the Friedmann equations, we find

$$H^2 \sim \rho \sim a^{-n(\omega)}, \tag{68}$$

where $n(\omega) = 3(1 + \omega)$ and the scale factor obeys:

$$a(t) = a_0 \begin{cases} t^{\frac{2}{n(\omega)}}, & \text{if } \omega \neq 1 \\ e^{Ht}, & \text{if } \omega = -1. \end{cases} \tag{69}$$

For all matter types other than $\omega = -1$, $H^2 \sim \rho \sim a^{-n(\omega)}$ where $a(t) = a_0 t^{\frac{2}{n(\omega)}}$. Then $\rho(t) \sim \left(a_0 t^{\frac{2}{n(\omega)}}\right)^{-n(\omega)} = a_0^{-n(\omega)} t^{-2}$ or $\rho(t) \sim t^{-2}$ which means the density becomes arbitrarily large at $t = 0$, corresponding to a singularity. Penrose and Hawking demonstrated that with a suitable definition of singularity, the world lines end at finite proper time $t = 0$ going backwards.

The equations of state $p = \omega_i \rho_i$, for ω_i constant and $i = 1, \ldots, \#$matter types, satisfy $H^2 \sim \rho \sim a^{-n(\omega)}$. The density parameter for matter of density ρ_i is

$$\Omega_i := \frac{8\pi G}{3} \frac{\rho_i}{H^2}. \tag{70}$$

However, this does not apply to the scalar curvature of the underlying time-like three-manifold slices or the cosmological constant, which are pseudo-matter types. These arise in the action as corrections to the matter Lagrangian:

$$\int_M d^4x \sqrt{-g}(R + (2\Lambda + \mathcal{L}_{\text{matter}})). \tag{71}$$

The pseudo-density parameter for spatial curvature is

$$\Omega_k = -\frac{k}{H^2 a^2}. \tag{72}$$

Likewise, the density parameter for the cosmological constant is

$$\Omega_\Lambda := \frac{\Lambda}{3H^2}. \tag{73}$$

Using the Hubble function and the density parameters together with the Friedmann equations, we obtain

$$\Omega_k + \sum_{\substack{i=1 \\ \text{real matter types}}}^{n} \Omega_i = 1 \tag{74}$$

for the matter types $i = 1, \ldots, n = \#$matter types. From the Friedmann acceleration equation, we ascertain the following identity.

$$H^{-2}\frac{\ddot{a}}{a} = -\frac{1}{2} \sum_{\substack{i=1 \\ \text{real matter types}}}^{n} (1 + 3\omega_i)\Omega_i \tag{75}$$

We consider the dominant matter types at various epochs. Since $\phi_i \sim a^{-n(\omega_i)}$ for any type of matter, i.e. radiation ($\omega = 1/3$) $n_\gamma = 4$, dust ($\omega = 0$) $n_M = 3$, spatial curvature ($\omega = -1/3$) $n_k = 2$, and dark energy ($\omega = -1$) $n_\Lambda = 0$, it follows that

$$\begin{aligned} a^2\Omega_k &= -\frac{ka^2}{H^2a^2} \sim \Omega_\Lambda \sim \frac{1}{H^2} \\ a^3\Omega_M &= \frac{8\pi G}{3a^3H^2}a^3 \sim \frac{1}{H^2} \\ a^4\Omega_\gamma &= \frac{8\pi G}{3a^4H^2}a^4 \sim \frac{1}{H^2}. \end{aligned} \tag{76}$$

Thus, we conclude that

$$\Omega_\Lambda \sim a^2\Omega_k \sim a^3\Omega_M \sim a^4\Omega_\gamma, \tag{77}$$

which dictates the distribution of matter types for different epochs.

Remark 5.3. *A vacuum energy-dominated universe is described by the metric*

$$ds^2 = -dt^2 + e^{Ht}(dx^2 + dy^2 + dz^2) \tag{78}$$

where the Hubble parameter H is constant. De Sitter space is the spacetime with positive cosmological constant. As $a \to \infty$, curvature and matter become negligible so the universe approaches a de Sitter space asymptotically.

The ΛCDM model is a concordance cosmology theory which posits that the universe contains dark energy Λ and cold dark matter (CDM). Such a model provides an argument for the large-scale distribution of galaxies as well as the accelerated expansion of the universe, as observed by Hubble in 1929 [9].

Let Ω_γ be the density parameter for radiation, Ω_k the density parameter for spatial curvature, Ω_Λ the density parameter for dark energy, and Ω_M the density parameter for real matter. The Friedmann equation may be re-written as

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = H_0^2 \left[\Omega_M \left(\frac{a(t_0)}{a(t)}\right)^3 + \Omega_\Lambda\right] \tag{79}$$

where H_0 is the present value of the Hubble constant and $a(t_0) = a_0 = 1$, by convention. The conformal time η may be expressed in terms of physical time t as $\eta = \frac{dt}{a(t)}$. Thus, we write the first Friedmann equation as [9]:

$$\frac{1}{a^2(t)} \left(\frac{da(t)}{dt}\right)^2 = H_0^2 \left[\Omega_M \frac{1}{a^3} + \Omega_\Lambda\right]. \tag{80}$$

Integrating the Friedmann equation, we obtain the expression:

$$H_0 t = \frac{2}{3\sqrt{\Omega_\Lambda}} \int_0^{u'} \frac{du}{\sqrt{1+u^2}} \tag{81}$$

where the upper bound is $u' = \sqrt{\frac{\Omega_\Lambda}{\Omega_M}} a^{3/2}$. The current age of the universe may be determined when $u' = \sqrt{\frac{\Omega_\Lambda}{\Omega_M}}$. As such, the age of the universe in terms of these parameters is

$$H_0 t_0 = \frac{2}{3\sqrt{\Omega_\Lambda}} \ln\left(\frac{1+\sqrt{\Omega_\Lambda}}{\sqrt{\Omega_M}}\right). \tag{82}$$

The scale factor may be analytically computed from Equation (81) as [6]:

$$a(t) = \left(\frac{\Omega_M}{\Omega_\Lambda}\right)^{1/3} \left[\sinh\left(\frac{3}{2}\sqrt{\Omega_\Lambda} H_0 t\right)\right]^{2/3}. \tag{83}$$

In the limiting case of $H_0 t \ll 1$, the contribution from the cosmological constant tends to zero and the universe expands at a decelerated rate [9]. During a matter-dominated epoch, we develop a Taylor series expansion of Equation (82):

$$a(t) = \left(\frac{9}{4}\Omega_M\right)^{1/3} (H_0 t)^{2/3}. \tag{84}$$

At present, the scale factor a_M is the ratio of the current radiation density to the current matter density; namely, the redshift of the current universe is

$$z_m = \frac{a_0}{a_M} - 1 = \frac{\Omega_M}{\Omega_\gamma} - 1. \tag{85}$$

According to measurements by the seven-year Wilkinson Microwave Anisotropy Probe (WMAP), the redshift is approximately $z_m = 3,196$ [10]. The physical time t_m, which corresponds to matter domination, is given by

$$\frac{t_m}{t_0} = \sqrt{\frac{\Omega_\Lambda}{\Omega_\lambda}} \frac{a_M^{3/2}}{\ln\left(\frac{1+\sqrt{\Omega_\Lambda}}{\sqrt{\Omega_M}}\right)}. \tag{86}$$

In the limiting case of $H_0 t \gg 1$, the cosmological constant contribution dominates while matter density is suppressed, according to Equation (77), which means that the scale factor becomes:

$$a(t) = \left(\frac{\Omega_M}{4\Omega_\Lambda}\right)^{1/3} \exp\left(\sqrt{\Omega_\Lambda} H_0 t\right), \tag{87}$$

corresponding to a de Sitter space. See Figure 3 for an illustration of $a(t)$ from $t = 0$ to $t = 0.02$ seconds in an FLRW universe, and for the cases of $H_0 t \ll 1$ and $H_0 t \gg 1$.

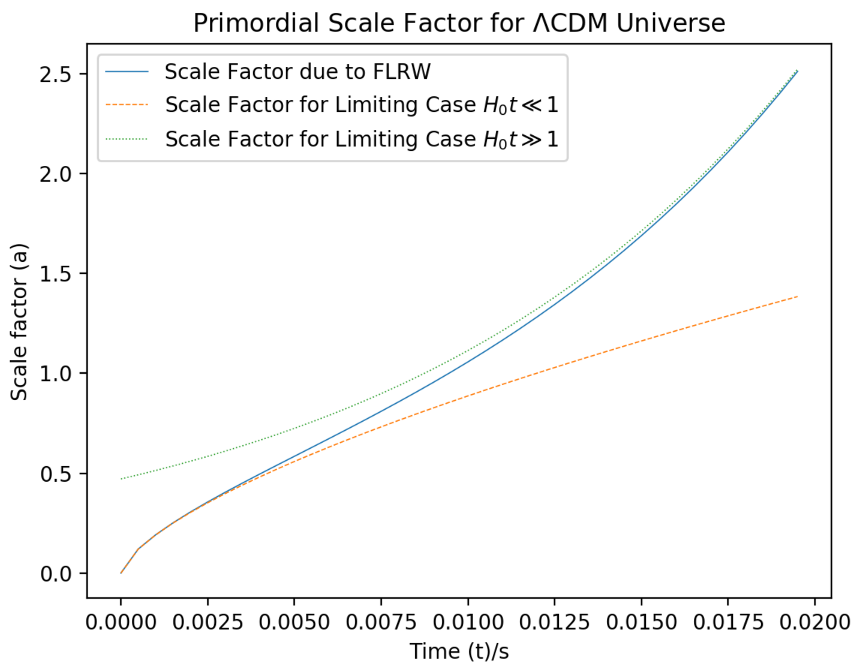

Figure 3: Primordial scale factor $a(t)$ from $t = 0$ to $t = 0.02$ seconds into the expansion of a ΛCDM universe due to the Friedmann–Lemaitre–Robertson–Walker metric and the limiting cases of $H_0 t \ll 1$ and $H_0 t \gg 1$.

To determine the transient condition for changing from a decelerated expansion to an accelerated expansion, we consider the second Friedmann equation. The equation may be expressed as [9]:

$$\frac{1}{a(t)} \left(\frac{d^2 a(t)}{dt^2}\right)^2 + \frac{1}{2}\left(\frac{\Omega_M}{a^3(t)} - 2\Omega_\Lambda\right) = 0.$$

When $t = t_\Lambda := \frac{2}{3H_0\sqrt{\Omega_\Lambda}}$, the declerated expansion transitions into an accelerated expansion determined by the second-order linear differential equation

$$\frac{d^2a(t)}{dt^2} = 0.$$

When $t = t_\Lambda$, the scale factor is

$$a_\Lambda = \left(\frac{\Omega_M}{2\Omega_\Lambda}\right)^{1/3}, \tag{88}$$

which has a corresponding redshift of

$$z_\Lambda = \frac{1 - a_\Lambda}{a_\Lambda} = \left(\frac{2\Omega_\Lambda}{\Omega_M}\right)^{1/3} - 1.$$

The physical time for which the expansion changes from a deceleration to an acceleration is determined by the relation

$$H_0 t_\Lambda = \int_0^{a_\Lambda} \frac{\sqrt{a}da}{\sqrt{\Omega_M + \Omega_\Lambda a^3}}, \tag{89}$$

whereby, according to WMAP and the Planck mission, $H_0 = 67.3\ \mathrm{km\,s^{-1}Mpc^{-1}}, \Omega_\gamma = 9.24 \times 10^{-5}, \Omega_M = 0.315, \Omega_\Lambda = 0.685, \Omega_k = 0$.

If ω is the equation of state for dark energy, the Friedmann equation may be written as

$$H(a) = \frac{\dot{a}}{a} = H_0\sqrt{\Omega_k a^{-2} + (\Omega_c + \Omega_b)a^{-3} + \Omega_\gamma a^{-4} + \Omega_\Lambda a^{-3(1+\omega)}} \tag{90}$$

where we denote by c cold dark matter, by γ radiation, by Λ dark energy, and by b baryons. For the ΛCDM six-parameter model, we assume $\Omega_k = 0$ and $\omega = -1$ so (see Figure 4)

$$H(a) = \sqrt{\Omega_\Lambda + \Omega_M a^{-3} + \Omega_\gamma a^{-4}}. \tag{91}$$

It follows that

$$t(a) = \int_0^a \frac{a' da'}{a'^2 H(a')} = \frac{1}{H_0}\int_0^a \frac{a' da'}{\sqrt{\Omega_\gamma + \Omega_M a' + \Omega_k a'^2 + \Omega_\Lambda a'^4}} \tag{92}$$

is the age of the universe as a function of a.

In the late epoch, $\Omega_\gamma \sim 10^{-4}$. By ignoring this term, the analytic solution of the aforementioned first-order differential equation (90) is [4]:

$$a(t) = \left(\frac{\Omega_M}{\Omega_\Lambda}\right)^{1/3} \sinh^{2/3}\left(\frac{3H_0\sqrt{\Omega_\Lambda}t}{2}\right).$$

The transition from a decelerating universe to an accelerating universe is the physical time for which $a = (\Omega_M/2\Omega_\Lambda)^{1/3}$. Thus, for a ΛCDM model of the universe $(M, \mathcal{O}_M, \mathcal{A}_{C^\infty(M)}, {}^{\mathrm{L.C.}}\nabla)$, radiation Ω_γ is negligible and spacetime is foliated as $M \cong \mathbb{R} \times \Sigma$ where Σ is the three-manifold with corresponding spatial curvature k chosen to be 0.

Recall that the Ricci scalar for the Friedmann–Lemaitre–Robertson–Walker metric is given by

$$R = 6\left(\frac{\ddot{a}(t)}{c^2 a(t)} + \frac{\dot{a}^2(t)}{c^2 a^2(t)} + \frac{k}{a^2(t)}\right).$$

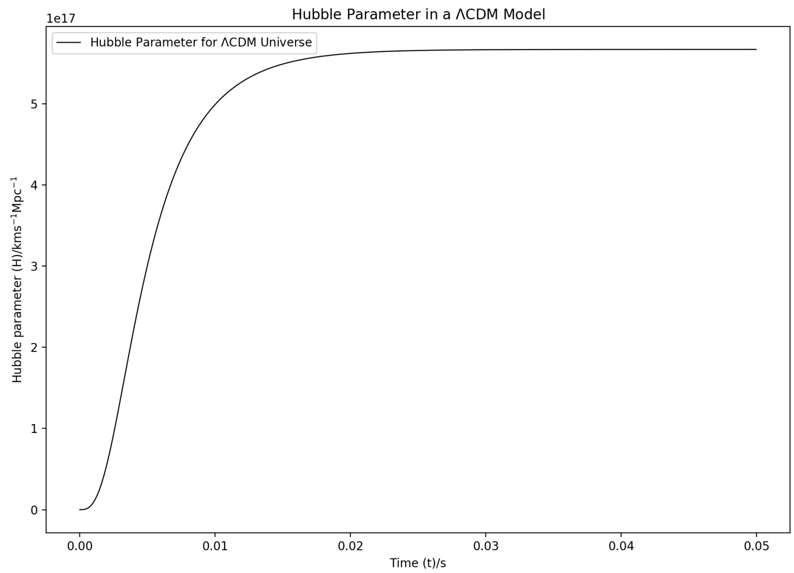

Figure 4: Primordial Hubble parameter $H(a)$ from $t = 0$ to $t = 0.05$ seconds into the expansion of a ΛCDM universe.

If $\omega = 0$ for a matter-dominated universe, then $\dot{a}(t) \propto \frac{2}{3}t^{-1/3}$, $\ddot{a}(t) \propto -\frac{2}{9}t^{-4/3}$ so

$$\begin{aligned} R_M(t) &= 6\left(\frac{1}{c^2}\left(\frac{-\frac{2}{9}t^{-4/3}}{\frac{2}{3}t^{-1/3}}\right) + \frac{1}{c^2}\left(\frac{\frac{2}{3}t^{-1/3}}{t^{2/3}}\right)^2 + k\left(t^{-2/3}\right)^2\right) \\ &= 6\left(-\frac{2}{9c^2}t^{-2} + \frac{4}{9c^2}t^{-2}\right) \\ &= \frac{4}{3}\frac{t^{-2}}{c^2} \sim O(t^{-2}) \end{aligned} \tag{93}$$

because $k = 0$ in a ΛCDM model. Likewise, if $\omega = 1/3$ for a radiation-dominated universe, then $\dot{a}(t) \propto \frac{1}{2}t^{-1/2}$, $\ddot{a}(t) \propto -\frac{1}{4}t^{-3/2}$ so

$$\begin{aligned} R_\gamma(t) &= 6\left(\frac{1}{c^2}\left(\frac{-\frac{1}{4}t^{-3/2}}{t^{1/2}}\right) + \frac{1}{c^2}\left(\frac{\frac{1}{2}t^{-1/2}}{t^{1/2}}\right)^2 + 0\right) \\ &= 6\left(-\frac{1}{4c^2t^2} + \frac{1}{4c^2t^2}\right) = 0. \end{aligned} \tag{94}$$

If $\omega = -1$ for a dark energy-dominated universe, then $a(t) \propto e^{Ht}$, $\dot{a}(t) \propto He^{Ht}$, $\ddot{a}(t) \propto H^2e^{Ht}$ so

$$\begin{aligned} R_\Lambda(t) &= 6\left(\frac{1}{c^2}\left(\frac{H^2e^{Ht}}{e^{Ht}}\right) + \frac{1}{c^2}\left(\frac{He^{Ht}}{e^{Ht}}\right)^2 + 0\right) \\ &= 6\left(\frac{H^2}{c^2} + \frac{H^2}{c^2}\right) \\ &= \frac{12H^2}{c^2}. \end{aligned} \tag{95}$$

If the matter in the universe is a mixture of non-interacting fluids then

$$\dot{\rho}_i = -3H\left(\rho_i + \frac{p_i}{c^2}\right)$$

for each fluid i whereby $\dot{\rho}_i = -3H(\rho_i + \omega_i\rho_i)$. It follows that $\rho_i \propto a^{-3(1+\omega_i)}$. For instance, if we consider a density given by a combination of dust A ($\omega = 0$), radiation B ($\omega = 1/3$), and dark energy C ($\omega = -1$) then the combined density is given by $\rho = Ca^0 + Aa^{-3} + Ba^{-4}$. We substitute this density into $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$ where $k = 0$ for a ΛCDM model to obtain

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\left(Ca^0 + Aa^{-3} + Ba^{-4}\right)$$

or

$$\left(\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}(Ca^2 + Aa^{-1} + Ba^{-4}).$$

Since $\Omega_\gamma \sim 0$, the analytic solution

$$a(t) = \left(\frac{\Omega_M}{\Omega_\Lambda}\right)^{1/3}\left[\sinh\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)\right]^{2/3}$$

may be differentiated iteratively. We obtain the following derivatives:

$$\begin{aligned}
\dot{a}(t) &= \left(\frac{\Omega_M}{\Omega_\Lambda}\right)^{1/3}\sqrt{\Omega_\Lambda}H_0\frac{\cosh\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)}{\left(\sinh(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t)\right)^{1/3}},\\
\ddot{a}(t) &= \frac{\left(\frac{\Omega_M}{\Omega_\Lambda}\right)^{1/3}\sqrt{\Omega_\Lambda}H_0}{\left(\sinh\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)\right)^{2/3}}\left[\frac{3}{2}\sqrt{\Omega_\Lambda}H_0\sinh\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)\left(\sinh\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)\right)^{1/3}\right.\\
&\quad\left.-\frac{1}{3}\left(\sinh\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)\right)^{-2/3}\cosh\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)\frac{3}{2}\sqrt{\Omega_\Lambda}H_0\cosh\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)\right].
\end{aligned}$$

The Hubble parameter is

$$\begin{aligned}
H &= \frac{\dot{a}}{a}\left(\frac{\Omega_M}{\Omega_\Lambda}\right)^{1/3}\sqrt{\Omega_\Lambda}H_0\frac{\cosh\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)}{\left(\sinh\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)\right)^{1/3}}\left(\frac{\Omega_M}{\Omega_\Lambda}\right)^{-1/3}\frac{1}{\left(\sinh\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)\right)^{2/3}}\\
&= \sqrt{\Omega_\Lambda}H_0\coth\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\ddot{a}(t)}{a(t)} &= \left(\frac{\Omega_M}{\Omega_\Lambda}\right)^{1/3}\sqrt{\Omega_\Lambda}H_0\left[\frac{3}{2}\sqrt{\Omega_\Lambda}H_0\left(\sinh\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)\right)^{4/3}-\frac{1}{2}\sqrt{\Omega_\Lambda}H_0\frac{\cosh^2\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)}{\left(\sinh\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)\right)^{2/3}}\right]\\
&\quad\times\frac{1}{\left(\sinh\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)\right)^{2/3}}\left(\frac{\Omega_M}{\Omega_\Lambda}\right)^{-1/3}\frac{1}{\left(\sinh\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)\right)^{2/3}}\\
&= \sqrt{\Omega_\Lambda}H_0\left[\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 - \frac{1}{2}\sqrt{\Omega_\Lambda}H_0\coth^2\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)\right]\\
&= \frac{1}{2}H_0^2\Omega_\Lambda\left(3 - \coth^2\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)\right).
\end{aligned}$$

Thus, since $k = 0$ in the ΛCDM model,

$$\begin{aligned} R(t) = R^{\mu\nu}(t)g_{\mu\nu}(t) &:= 6\left(\frac{\ddot{a}(t)}{c^2 a(t)} + \frac{\dot{a}^2(t)}{c^2 a^2(t)}\right) \\ &= 6\left(\frac{1}{2c^2}H_0^2\Omega_\Lambda\left(3 - \coth^2\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)\right) + \frac{1}{c^2}\Omega_\Lambda H_0^2\coth^2\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)\right) \\ &= \frac{9H_0^2\Omega_\Lambda}{c^2} + \frac{3H_0^2\Omega_\Lambda}{c^2}\coth^2\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right) \end{aligned} \tag{96}$$

is the time-evolving Ricci scalar curvature for the ΛCDM universe under the assumption that the radiation density parameter is negligible at late times, for $\Omega_\Lambda \sim a^2\Omega_k \sim a^3\Omega_M \sim a^4\Omega_\gamma$ (see Figure 5).

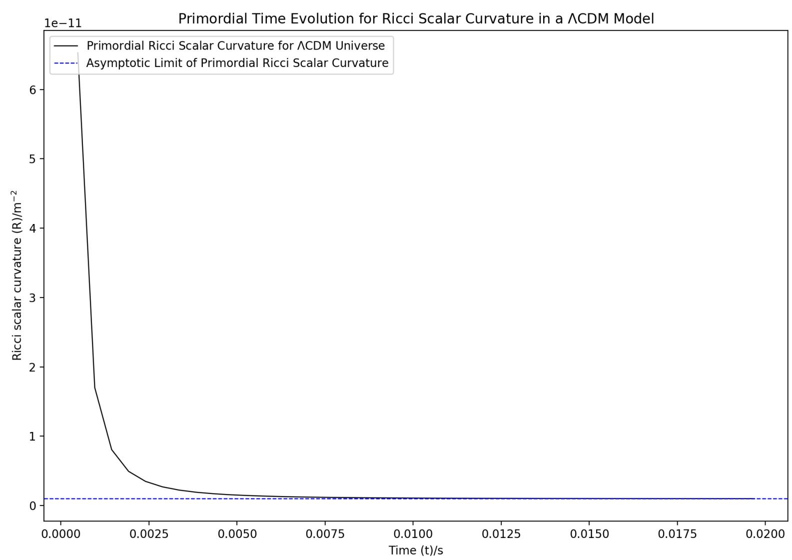

Figure 5: Primordial Ricci scalar curvature $R(t)$ from $t = 0$ to $t = 0.1$ seconds into the expansion of a ΛCDM universe.

The asymptotic limit of the Ricci scalar curvature $R(t)$ for infinite time is

$$\begin{aligned} R_\infty := \lim_{t\to\infty} R(t) &= \lim_{t\to\infty} \frac{9H_0^2\Omega_\Lambda}{c^2} + \frac{3H_0^2\Omega_\Lambda}{c^2}\coth^2\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right) \\ &= \frac{9H_0^2\Omega_\Lambda}{c^2} + \frac{3H_0^2\Omega_\Lambda}{c^2}\lim_{t\to\infty}\coth^2\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right) \\ &= \frac{12H_0^2\Omega_\Lambda}{c^2}. \end{aligned} \tag{97}$$

We use the Laurent expansion of hyperbolic cotagent to approximate the hyperbolic expression in Equation (96) in an open neighborhood of $t = 0$ as (see Appendix C).

$$\coth\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right) = \frac{2}{3}\frac{1}{\sqrt{\Omega_\Lambda}H_0}\frac{1}{t} + \frac{1}{2}\sqrt{\Omega_\Lambda}H_0 t - \frac{9}{40}\Omega_\Lambda^{3/2}H_0^3 t^3 + O(t^5). \tag{98}$$

Thus, up to first-order approximation and for sufficiently small times t, we have the asymptotic relation $\coth\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right) \sim \frac{2}{3}\frac{1}{\sqrt{\Omega_\Lambda}H_0}\frac{1}{t}$ so that the primordial Ricci scalar curvature is:

$$\begin{aligned} R(t) &= \frac{9H_0^2\Omega_\Lambda}{c^2} + \frac{3H_0^2\Omega_\Lambda}{c^2}\left(\frac{2}{3}\frac{1}{\sqrt{\Omega_\Lambda}H_0}\frac{1}{t} + \frac{1}{2}\sqrt{\Omega_\Lambda}H_0 t - \frac{9}{40}\Omega_\Lambda^{3/2}H_0^3 t^3 + O(t^5)\right)^2 \\ &= \frac{9H_0^2\Omega_\Lambda}{c^2} + \frac{3H_0^2\Omega_\Lambda}{c^2}\left(\frac{2}{3}\frac{1}{\sqrt{\Omega_\Lambda}H_0}\frac{1}{t} + O(t)\right)^2 \\ &= \frac{9H_0^2\Omega_\Lambda}{c^2} + \frac{4}{3}\frac{1}{c^2t^2} + O(1). \end{aligned} \tag{99}$$

By approximating hyperbolic cosine in an open neighborhood of $t = 0$, as in Equation (98), we can carry out the following computation.

$$\begin{aligned} R(t) &= \frac{9H_0^2\Omega_\Lambda}{c^2} + \frac{3H_0^2\Omega_\Lambda}{c^2}\left(\frac{2}{3}\frac{1}{\sqrt{\Omega_\Lambda}H_0}\frac{1}{t} + \frac{1}{2}\sqrt{\Omega_\Lambda}H_0 t - \frac{9}{40}\Omega_\Lambda^{3/2}H_0^3 t^3 + O(t^5)\right)^2 \\ &= \frac{9H_0^2\Omega_\Lambda}{c^2} + \frac{3H_0^2\Omega_\Lambda}{c^2}\left(\frac{4}{9}\frac{1}{\Omega_\Lambda H_0^2}\frac{1}{t^2} + \frac{2}{3} - \frac{1}{20}H_0^2\Omega_\Lambda t^2 - \frac{9}{40}H_0^4\Omega_\Lambda^2 t^4 + \frac{81}{1600}H_0^6\Omega_\Lambda^3 t^6 + O(t^8)\right) \\ &= \frac{4}{3}\frac{1}{c^2t^2} + 11\frac{H_0^2\Omega_\Lambda}{c^2} - \frac{3}{20}\frac{H_0^4\Omega_\Lambda^2}{c^2}t^2 - \frac{27}{40}\frac{H_0^6\Omega_\Lambda^6}{c^2}t^4 + \frac{243}{1600}\frac{H_0^8\Omega_\Lambda^4}{c^2}t^6 + O(t^8). \end{aligned} \tag{100}$$

6 de Sitter Space, Local Gauge Theory, and Non-Minimally Coupled Inflation

At infinite time, dark energy will dominate the density ρ of the universe. If the universe $M = \mathbb{R}\times\Sigma$ is spatially homogeneous and isotropic, the Copernican principal implies that the spatial three-manifold Σ is maximally symmetric. Isotropy and homogeneity together imply that spacetime has the maximum number of Killing vectors allowed, spanning a Lie algebra. The solution to the non-vacuum Einstein field equations with $\Lambda > 0$ corresponds to a de Sitter space, the maximally symmetric spacetime with positive curvature. Likewise, the solution with $\Lambda < 0$ represents an anti-de Sitter space, which is also maximally symmetric with negative curvature. The late epoch, present-day value of the cosmological constant in the ΛCDM universe is $\Lambda = 1.1056\times 10^{-52}\text{m}^{-2}$ [1]. Therefore, we adopt the model of de Sitter space. The field that gives rise to inflation is called the inflaton. In an expanding spacetime, two inertial observers move farther apart with accelerating velocity following the inside-out black hole polar metric [8]:

$$-(1-\Lambda r^2)dt^2 + \frac{1}{1-\Lambda r^2}dr^2 + r^2 d\Omega^2. \tag{101}$$

Such an exponentially expanding spacetime is called a de Sitter space, filled everywhere by a vacuum energy proportional to Λ and devoid of matter and radiation. Thus, de Sitter space is the limiting case of the standard FLRW model of inflation. The scale factor associated with a de Sitter space is $a(t) = a_0e^{Ht}$ where $H \propto \sqrt{\Lambda}$. The Friedmann–Lemaitre–Robertson–Walker metric in hyperpolar coordinates is

$$\begin{aligned} ds^2 &= c^2dt^2 - e^{Ht}[dr^2 + r^2 d\Omega_2^2] \\ &= c^2dt^2 - e^{Ht}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \end{aligned} \tag{102}$$

for $d\Omega_2^2$ the metric on the round two-sphere.

The Riemann curvature tensor of a maximally symmetric n-dimensional manifold is given by

$$R_{\mu\nu\rho\sigma} = k(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \tag{103}$$

where k is the normalized Ricci curvature

$$k = \frac{R}{n(n-1)} \tag{104}$$

with constant Ricci scalar R [3]. For vanishing curvature k, the maximally symmetric spacetime is a Minkowski spacetime $\mathbb{R}^{1,n}$ with metric

$$ds^2 = -dx_0^2 + \sum_{i=1}^{n} dx_i^2. \tag{105}$$

De Sitter space dS_n corresponds to the maximally symmetric spacetime with positive curvature k. In fact, de Sitter space is a Lorentzian sub-manifold of the hyperboloid

$$-x_0^2 + \sum_{i=1}^{n} x_i^2 = \alpha^2 \tag{106}$$

for $\alpha \neq 0$. The metric on dS_n is pulled back from the metric of the ambient Minkowski space $\mathbb{R}^{1,n}$. De Sitter space dS_n is topologically equivalent to $\mathbb{R} \times S^{n-1}$ as the quotient of orthogonal groups $\frac{O(1,n)}{O(1,n-1)}$ with isometry group $O(1,n)$. Thus, the metric has $n(n+1)/2$ Killing vector fields, whereby the Riemann curvature tensor is

$$R_{\mu\nu\rho\sigma} = \frac{1}{\alpha^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}).$$

Since the Ricci tensor is proportional to the metric tensor, i.e. $R_{\mu\nu} = \frac{n-1}{\alpha^2}g_{\mu\nu}$, de Sitter space is an Einstein manifold. As such, it is a vacuum solution with cosmological constant

$$\Lambda = \frac{(n-1)(n-2)}{2\alpha^2} \tag{107}$$

and Ricci scalar curvature

$$R = \frac{n(n-1)}{\alpha^2}. \tag{108}$$

We use closed slicing on a patch of four-dimensional de Sitter sapce dS_4 to induce local coordinates (t, ξ, θ, ϕ) given by [3]:

$$\begin{aligned} x_0 &= \alpha \sinh(t/\alpha) \\ x_1 &= \alpha \cosh(t/\alpha) \sin(\xi) \cos(\theta) \\ x_2 &= \alpha \cosh(t/\alpha) \sin(\xi) \sin(\theta) \cos(\phi) \\ x_3 &= \alpha \cosh(t/\alpha) \sin(\xi) \sin(\theta) \sin(\phi) \\ x_4 &= \alpha \cosh(t/\alpha) \cos(\xi), \end{aligned} \tag{109}$$

from which we derive the nondegenerate Lorentzian metric:

$$\begin{aligned} ds^2 &= -dt^2 + \alpha^2 \cosh^2(t/\alpha) d\Omega_3^2 \\ &= -dt^2 + \alpha^2 \cosh^2(t/\alpha) \left[d\xi^2 + \sin^2\xi(d\theta^2 + \sin^2\theta d\phi^2)\right] \end{aligned} \tag{110}$$

for $d\Omega_3^2 = d\xi^2 + \sin^2\xi d\Omega_2^2$ the round metric on the three-sphere, given in terms of the round metric on the two-sphere by $d\Omega_2^2$. Under a conformal transformation $\tan(\chi/2) = \tanh(t/2\alpha)$, we obtain the conformally equivalent metric [11]:

$$ds^2 = \frac{\alpha^2}{\cos^2\chi}(-d\chi^2 + d\Omega_3^2). \tag{111}$$

Thus, $\Lambda = 3/\alpha^2$ and $R = 4\Lambda$. We henceforth assume $\alpha = 1$, corresponding to the positive spatial curvature $k = 1$. The spatial sub-manifold of four-dimensional de Sitter space is invariant under rotations of $SO(3)$ which gives rise to a symmetry and, by Noether's theorem, a conservation law, namely that of the energy momentum tensor: $T^{\mu\nu}_{;\nu} = 0$. Let Φ be the inflation potential for de Sitter space.

Consider the model for non-minimally coupled inflation in which the coupling constant ξ between gravity R and the magnitude of the inflaton field ϕ is not negligible, with an associated action [5]:

$$\begin{aligned} S[\phi, R] &= \int_M d^4x\sqrt{-g}\mathcal{L}(\phi, g) \\ &= \int_M d^4x\sqrt{-g}\left[\frac{m_P^2}{2}R - \frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - V(\phi) - \frac{\xi}{2}R\phi^2\right], \end{aligned} \tag{112}$$

where $m_P = \sqrt{\hbar c/G}$ is the Planck mass and $V(\phi)$ is the potential as a function of the inflaton field. The improved Noether current $j^\mu(\mathcal{L}, X) = T^{\mu\nu}X_\mu$, associated with the Killing symmetry X of the action functional, must be linear in X and depend on the values of X. Diffeomorphism invariance of the functional means $\delta_X S[\phi, g] = 0$ for all X, ϕ, g. Introduce the vector of fields $\Phi = (\phi, R)^T$. We then define the globally gauge invariant Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{glb}} &= \frac{1}{2}(\partial_\mu\Phi)^T\partial^\mu\Phi - \frac{1}{2}m^2\Phi^T\Phi \\ &= \frac{m_P^2}{2}R - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi) - \frac{\xi}{2}R\phi^2 \end{aligned} \tag{113}$$

where $m := \int_M d^4x\sqrt{-g}\rho(x)$ is the mass of the universe for spacetime density $\rho(x)$, $\partial_\mu\phi$ is the partial derivative of ϕ in each of the four dimensions, and $R(t) = \frac{9H_0^2\Omega_\Lambda}{c^2} + \frac{3H_0^2\Omega_\Lambda}{c^2}\coth^2\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0t\right)$. The Lagrangian is invariant under the gauge group transformation $\Phi \mapsto \widehat{\Phi} = \Pi\Phi$ for $\Pi \in O(2)$ a constant representative. Thus, the structure group preserves the global symmetry of $\mathcal{L}_{\text{glb}}$. If we impose the condition that the Lagrangian must be locally $O(2)$-invariant, then $\Pi \in O(2)$ should be a function of spacetime $\Pi := \Pi(\mathbf{x})$, $\mathbf{x} \in M$. If ϵ^α are the generators of $SO(2)$, then Noether's theorem implies the conservation of currents $j^\alpha_\nu = i\partial_\nu\Phi^T\epsilon^\alpha\Phi$ [5]. We define the gauge covariant derivative of the metric-induced Levi-Civita connection as

$$\nabla_\mu := \partial_\mu - i\xi A_\mu. \tag{114}$$

Consider the locally gauge invariant Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{loc}} &= \frac{1}{2}(\nabla_\mu\Phi)^T\nabla^\mu\Phi - \frac{1}{2}m^2\Phi^T\Phi \\ &= \frac{m_P^2}{2}R - \frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - V(\phi) - \frac{\xi}{2}R\phi^2. \end{aligned} \tag{115}$$

The Lagrangian has local $O(2)$ guage group-invariance, preserved under the transformation $\Phi \mapsto \widehat{\Phi} = \Pi\Phi$, because the covariant derivative transforms identically as $\nabla_\mu\Phi \mapsto \widehat{\nabla_\mu\Phi} = \Pi(\nabla_\mu\Phi)$. The gauge field $A(\mathbf{x})$, a function of spacetime, transforms as

$$\widehat{A_\mu} = \Pi A_\mu \Pi^{-1} + \frac{i}{\xi}(\partial_\mu \Pi)\Pi^{-1}. \tag{116}$$

It is a member of a Lie algebra and may therefore be expressed as

$$A_\mu = \sum_\nu A_\mu^\nu \epsilon^\nu, \tag{117}$$

which means there are as many generators of the Lie algebra as there are gauge fields. It follows that

$$\begin{aligned}\Pi A_\mu (I - \Pi^{-1}) &= \frac{i}{\xi}(\partial_\mu \Pi)\Pi^{-1},\\ A_\mu &= \frac{i}{\xi}\Pi^{-1}(\partial_\mu \Pi)\Pi^{-1}(I - \Pi^{-1})^{-1}.\end{aligned} \tag{118}$$

In general, the variation of $S[\phi, R]$ with respect to an infinitesimal diffeomorphism X on M is

$$\delta_X S[\phi, g] = \int_M d^4x\sqrt{-g}\left(\frac{\delta\mathcal{L}(\phi, g)}{\delta g_{\mu\nu}}\delta_X g_{\mu\nu} + \frac{\delta\mathcal{L}(\phi, g)}{\delta\phi^j}\delta_X\phi^j + \nabla_\mu(T^{\mu\nu}X_\nu)\right). \tag{119}$$

Thus, we vary the action for non-minimally coupled inflation with respect to the inverse metric:

$$\begin{aligned}\delta S[\phi, R] &= \delta\int_M d^4x\sqrt{-g}\left[\frac{m_P^2}{2}g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla^\nu\phi - V(\phi) - \frac{\xi}{2}g^{\mu\nu}R_{\mu\nu}\phi^2\right]\\ &= \int_M d^4x\Bigg[\sqrt{-g}\left(\frac{m_P^2}{2}R_{\mu\nu} - \frac{1}{2}\nabla_\mu\phi\nabla^\nu\phi - \frac{\xi}{2}R_{\mu\nu}\phi^2\right)\delta g^{\mu\nu}\\ &\quad + \Big(-\frac{1}{2}g_{\mu\nu}\Big)(\sqrt{-g}\delta g^{\mu\nu})\left(\frac{m_P^2}{2}g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla^\nu\phi - V(\phi) - \frac{\xi}{2}g^{\mu\nu}R_{\mu\nu}\phi^2\right)\Bigg] = 0.\end{aligned} \tag{120}$$

We hence find the energy-momentum tensor via

$$\begin{aligned}T_{\mu\nu} &= -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}\\ &= -2\left[\frac{m_P^2}{2}R_{\mu\nu} - \frac{1}{2}\nabla_\mu\phi\nabla^\nu\phi - \frac{\xi}{2}R_{\mu\nu}\phi^2 - \frac{1}{2}g_{\mu\nu}\Big(\frac{m_P^2}{2}g^{\rho\sigma}R_{\rho\sigma} - \frac{1}{2}g^{\rho\sigma}\nabla_\rho\phi\nabla^\sigma\phi - V(\phi) - \frac{\xi}{2}g^{\rho\sigma}R_{\rho\sigma}\phi^2\Big)\right]\\ &= -m_P^2R_{\mu\nu} + \nabla_\mu\phi\nabla^\nu\phi + \xi R_{\mu\nu}\phi^2 + \frac{1}{2}m_P^2 g_{\mu\nu}R - \frac{1}{2}g_{\mu\nu}\nabla^\rho\phi\nabla^\sigma\phi + g_{\mu\nu}V(\phi) + \frac{1}{2}\xi g_{\mu\nu}R\phi^2.\end{aligned} \tag{121}$$

We apply the conservation law $\nabla^\nu T_{\mu\nu} = 0$ to conclude that

$$\begin{aligned}&- m_P^2\nabla^\nu R_{\mu\nu} + \nabla^\nu\nabla_\mu\phi\nabla^\nu\phi + \xi\nabla^\nu R_{\mu\nu}\phi^2 + \frac{1}{2}m_P^2\nabla^\nu g_{\mu\nu}R\\ &- \frac{1}{2}\nabla^\nu g_{\mu\nu}\nabla^\rho\phi\nabla^\sigma\phi + \nabla^\nu g_{\mu\nu}V(\phi) + \frac{1}{2}\xi\nabla^\nu g_{\mu\nu}R\phi^2 = 0.\end{aligned} \tag{122}$$

Recall $R_{\mu\nu} = 3g_{\mu\nu}$ and $R = 4\Lambda$. The equations of motion for the expanding ΛCDM universe are thus:

$$\begin{aligned}&\nabla^\nu \nabla_\mu \phi \nabla^\nu \phi + 3\xi \nabla^\nu g_{\mu\nu} \phi^2 + \frac{1}{2} m_P^2 \nabla^\nu g_{\mu\nu} R + \nabla^\nu g_{\mu\nu} V(\phi) + \frac{1}{2} \xi \nabla^\nu g_{\mu\nu} R \phi^2 \\ &= 3m_P^2 \nabla^\nu g_{\mu\nu} + \frac{1}{2} \nabla^\nu g_{\mu\nu} \nabla^\rho \phi \nabla^\sigma \phi\end{aligned} \tag{123}$$

where $R(t) = \frac{9H_0^2\Omega_\Lambda}{c^2} + \frac{3H_0^2\Omega_\Lambda}{c^2}\coth^2\left(\frac{3}{2}\sqrt{\Omega_\Lambda} H_0 t\right) = 4\Lambda(t) > 0$. While dark energy $\Lambda(t)$ is time-dependent in this model of coupled-inflation, it is strictly positive and, hence, the universe expands indefinitely since the universal density is slightly below the critical density $\rho_{\text{crit}} = \frac{3H_0^2}{8\pi G}$. That is, $\Omega = \rho/\rho_{\text{crit}} < 1$ because the spatial sections of de Sitter space are open. We conclude by determining the interaction Lagrangian, the difference of the locally gauge-invariant Lagrangian $\mathcal{L}_{loc} = \frac{1}{2}(\nabla_\mu \Phi)^T \nabla^\mu \Phi - \frac{1}{2} m^2 \Phi^T \Phi$ and the globally gauge-invariant Lagrangian $\mathcal{L}_{glb} = \frac{1}{2}(\partial_\mu \Phi)^T \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^T \Phi$. We expand the covariant derivative of the vector of scalar fields in an appropriate basis:

$$\begin{aligned}\nabla_\mu \Phi &= \nabla_{\frac{\partial}{\partial x^\mu}} \left(\Phi^\nu \frac{\partial}{\partial x^\nu} \right) \\ &= \left(\nabla_{\frac{\partial}{\partial x^\mu}} \Phi^\nu \right) \frac{\partial}{\partial x^\nu} + \Phi^\nu \left(\nabla_{\frac{\partial}{\partial x^\mu}} \frac{\partial}{\partial x^\nu} \right) = \frac{\partial \Phi^\nu}{\partial x^\mu} \frac{\partial}{\partial x^\nu} + \Phi^\nu \Gamma^\lambda_{\nu\mu} \frac{\partial}{\partial x^\lambda} \\ &= \partial_\mu \Phi + \Phi^\nu \Gamma^\lambda_{\nu\mu} \frac{\partial}{\partial x^\lambda} \\ &= \partial_\mu \Phi - i\xi A_\mu,\end{aligned}$$

from which we find that the potential is

$$A_\mu = i \frac{1}{\xi} \Phi^\nu \Gamma^\lambda_{\nu\mu} \frac{\partial}{\partial x^\lambda}. \tag{124}$$

The potential A_ν is a covector field such that $\partial_\mu A_\nu = G_{\mu\nu}$ is the Einstein tensor or $A_\nu = dx^\mu(G_{\mu\nu})$. Then $\xi dx^\mu(G_{\mu\nu}) = i\Phi^\nu \Gamma^\lambda_{\nu\mu} \frac{\partial}{\partial x^\lambda}$. We define the trace of the Einstein tensor to be $G = \text{Tr}(G_{\mu\nu} g^{\mu\nu})$ such that the coupling constant is

$$\xi = \frac{i}{G} \Phi^\nu \Gamma^\lambda_{\nu\mu} \frac{\partial}{\partial x^\lambda} \frac{\partial}{\partial x^\mu} g^{\mu\nu}. \tag{125}$$

It follows that the interaction Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{int}} &= \mathcal{L}_{\text{glb}} - \mathcal{L}_{loc} \\ &= i\frac{\xi}{2} \Phi^T A_\mu^T \partial^\mu \Phi + i\frac{\xi}{2} \left(\partial_\mu \Phi\right)^T A^\mu \Phi - \frac{\xi^2}{2} \left(A_\mu \Phi\right)^T A^\mu \Phi \\ &= -\frac{1}{2} \Phi^T \left(\Phi^\nu \Gamma^\lambda_{\nu\mu} \frac{\partial}{\partial x^\lambda} \right)^T \partial^\mu \Phi - \frac{1}{2} (\partial_\mu \Phi)^T \left(\Phi_\nu \Gamma^{\nu\mu}_{\lambda} \frac{\partial}{\partial x_\lambda} \right) \Phi \\ &+ \frac{1}{2} \left(\Phi^\nu \Gamma^\lambda_{\nu\mu} \frac{\partial}{\partial x^\lambda} \right)^T \left(\Phi_\nu \Gamma^{\nu\mu}_{\lambda} \frac{\partial}{\partial x_\lambda} \right) \Phi.\end{aligned}$$

7 Conclusion

We have mathematically derived the explicit time-dependence for Ricci scalar curvature and the expansion history of a Lambda-Cold Dark Matter universe in the late epoch with negligible radiation density parameter Ω_γ and large dark energy density parameter Ω_Λ. Namely, we have shown scalar curvature to evolve in time as a quadratic-hyperbolic form $R(t) = \frac{9H_0^2\Omega_\Lambda}{c^2} + \frac{3H_0^2\Omega_\Lambda}{c^2}\coth^2\left(\frac{3}{2}\sqrt{\Omega_\Lambda}H_0 t\right)$ with asymptotic limit $R_\infty = \frac{12H_0^2\Omega_\Lambda}{c^2}$. Hence, we determined the Euler–Lagrange equations of motion for non-minimally coupled inflation on a spacetime $\mathbb{R}\times\Sigma$ with a maximally symmetric spatial three-manifold Σ of constant positive curvature.

Appendix A Primer on Differential Geometry

We introduce the rudiments of bundle theory to assist the construction of the atlas on the tangent space of M.

Definition A.1. *A bundle is a triple (E, π, M) where the total space E is a smooth manifold (sometimes bundle), the base space M is a smooth manifold, and the projection map $\pi : E \to M$ is surjective.*

The construction of a C^∞-atlas on TM from the C^∞-atlas $\mathcal{A}$ on M is as follows. For the triple $(TM, \mathcal{O}_{TM}, \mathcal{A}_{TM})$ to be considered a smooth topological manifold, we must ensure the following.

(i) On the set-theoretic level, we define the tangent bundle as the disjoint union $TM := \bigsqcup_{pi \in M} T_pM$.

(ii) Construct the projection map

$$\begin{aligned} \pi :& TM \to M \\ & X \mapsto p \end{aligned}$$

where $p \in M$ is the unique point such that $X \in T_pM$. The projection map is surjective because we considered all tangent spaces in $TM = \bigsqcup_{p \in M} T_pM$. We must make TM a smooth manifold in order to determine whether the surjection is smooth.

(iii) Construct a topology on TM, inherited from M, that is the coarsest topology such that π becomes continuous, i.e. it is the initial topology with respect to π:

$$\mathcal{O}_{TM} := \{\text{preim}_\pi(\mathcal{U}) | \mathcal{U} \in \mathcal{O}\},$$

an endowed topology for $\mathcal{O}$ a subset of the power set $\wp(M)$. Thus, we have equipped TM with a topology $\mathcal{O}_{TM}$.

Let $\mathcal{A}_{TM} := \{(T\mathcal{U}, \xi_x) | (\mathcal{U}, x) \in \mathcal{A}\}$ where

$$\begin{aligned} \xi_x :& T\mathcal{U} \to \mathbb{R}^{2\dim M} \\ & X \mapsto ((x^1 \circ \pi)(X), \ldots, (x^{\dim M} \circ \pi)(X), (dx^1)_{\pi(X)}(X), \ldots, (dx^{\dim M})_{\pi(X)}(X)) \end{aligned} \tag{126}$$

where the elements $((x^1 \circ \pi)(X), \ldots, (x^{\dim M} \circ \pi)(X))$ contain information of $(\mathcal{U}, x)$ coordinates of $\pi(M)$, the elements $(x^{\dim M} \circ \pi)(X), (dx^1)_{\pi(X)}(X), \ldots, (dx^{\dim M})_{\pi(X)}(X))$ contain information of the components of the vector with respect to the chosen chart, and $(x^i \circ \pi)(X)$ is the i-th coordinate of $\pi(X) \in M$. The vector $X \in T_{\pi(X)}M$ is a vector in the tangent space to its own base point $\pi(X)$, which has components with respect to a basis due to the chart x:

$$X = X^i_{(x)} \left(\frac{\partial}{\partial x^i} \right)_{\pi(X)}. \tag{127}$$

Recall that the gradient of a smooth function $f \in C^\infty(M)$ is defined by

$$\begin{aligned} (df)_p :& T_pM \xrightarrow{\sim} \mathbb{R} \\ & X \mapsto (df)_p(X) = Xf \end{aligned}$$

so that $(dx^j)_{\pi(X)}(X) = (dx^j)_{\pi(X)} \left(X^i_{(x)} \left(\frac{\partial}{\partial x^i} \right)_{\pi(X)} \right) = X^i_{(x)} \left((dx^j)_{\pi(X)} \left(\frac{\partial}{\partial x^i} \right) \right)$.

Definition A.2. *A smooth vector field* χ *is a smooth map, i.e. a smooth section* $\Gamma(TM) = \{\chi : M \to TM | \textit{smooth section}\}$*, such that* $\pi \circ \chi := id_M$.

The vector space $(C^\infty(M), +, \cdot)$ is a collection of all smooth functions $C^\infty(M) := \{f : M \to \mathbb{R} | f \text{ smooth}\}$ satisfying the axioms of a ring. The $C^\infty(M)$-module $\Gamma(TM) = \{\chi : M \to TM | \text{smooth section}\}$ is a set that can be equipped with $\oplus$ and $\otimes$ operations:

$$(\chi \oplus \tilde{\chi})(f) := (Xf) +_{C^\infty(M)} \tilde{X}f \tag{128}$$

for $\chi_p \in T_pM$, whereby

$$\begin{aligned} \chi : &M \to TM \\ &p \mapsto \chi(p) \end{aligned}$$

acts on f by

$$\begin{aligned} \chi f : &M \to \mathbb{R} \\ &p \mapsto \chi(p)f \end{aligned}$$

and, for $g \in C^\infty(M)$,

$$(g \otimes \chi)(f) := g \cdot_{C^\infty(M)} \chi(f). \tag{129}$$

If we use $(g \otimes \chi)(f) := g \cdot_{C^\infty(M)} \chi(f)$ to define the s-multiplication, then $(\Gamma(TM), \oplus, \otimes)$ would be a $C^\infty(M)$-vector space, like a $\mathbb{R}$-vector space, which is known as a *module*. That is, $(\Gamma(TM), \oplus, \otimes)$, a vector space over a ring, is a $C^\infty(M)$-module.

The $C^\infty(M)$-module $\Gamma(TM) := \{\chi : M \to TM | \text{ smooth sections}\}$ is the set of smooth sections $\chi : M \to TM$ or the set of vector fields and $\Gamma(T^*M) := \{\chi : M \to T^*M | \text{ smooth sections}\}$ is the set of smooth sections $\chi : M \to T^*M$ or the set of covector fields.

Definition A.3. *An* (r, s)*-tensor field* T *is a* $C^\infty(M)$ *multi-linear map*

$$T : \Gamma(T^*M) \times \cdots \times \Gamma(T^*M) \times \Gamma(TM) \times \cdots \times \Gamma(TM) \xrightarrow{\sim} C^\infty(M).$$

Define a linear map

$$\begin{aligned} df : &\Gamma(TM) \xrightarrow{\sim} C^\infty(M) \\ &\chi \mapsto df(\chi) = \chi f. \end{aligned}$$

We can check that df is C^∞-linear, i.e. $df : \Gamma(TM) \xrightarrow{\sim} C^\infty(M)$ whereby $df(g\chi) = \chi g f = g(\chi f) = g df(\chi)$ so df is a $(0, 1)$-tensor field.

A vector field X can be used to provide a directional derivative Xf of a smooth function $f \in C^\infty(M)$. We use the notation $\nabla_X f := Xf$ to denote the directional derivative of f in the direction of X for $f \in C^\infty(M)$. Thus, $Xf = (df)(X) = \nabla_X f$. Since $X : M \to TM$, $X : C^\infty(M) \to C^\infty(M)$ then $df : \Gamma(TM) \to C^\infty(M)$ and $\nabla_X : C^\infty(M) \to C^\infty(M)$. We enumerate the properties for which ∇_X acting on a tensor field should necessarily satisfy. Any remaining freedom in choosing ∇ will be encoded as additional structure beyond $(M, \mathcal{O}, \mathcal{A})$.

Definition A.4. *A connection (alternatively covariant derivative and affine connection)* ∇ *on a smooth manifold* $(M, \mathcal{O}, \mathcal{A})$ *is a map that takes a pair consisting of a vector (field)* X *and a* (p, q)*-tensor field* T *and maps them to a* (p, q)*-tensor (field)* $\nabla_X T$ *satisfying:*

(i) (Extension rule.) $\nabla_X f = Xf$*, where* $f \in C^\infty(M)$ *is a* $(0, 0)$*-tensor field.*

(ii) *(Additivity rule.)* $\nabla_X(T+S) = \nabla_X T + \nabla_X S$, *for* T, S *both* (p,q)*-tensor fields.*

(iii) *(Leibnitz rule.)* $\nabla_X(T(\omega, Y)) = (\nabla_X T)(\omega, Y) + T(\nabla_X \omega, Y) + T(\omega, \nabla_X Y)$ *for* $\omega \in \Gamma(T^*M)$ *a covector field and* $Y \in \Gamma(TM)$ *a vector field.*

(iv) *(*$C^\infty(M)$*-linearity rule.)* $\nabla_{fX+Z}T = \nabla_{fX}T + \nabla_Z T = f\nabla_X T + \nabla_Z T$, *for all* $f \in C^\infty(M)$.

Definition A.5. *A manifold with connection (affine manifold) is a quadruple of structures* $(M, \mathcal{O}, \mathcal{A}, \nabla)$.

We remark that if $\nabla_X \bullet$ is the extension of the action of $X\bullet$ then $\nabla_\bullet \bullet$ is the extension of the action of d such that $\nabla_X \bullet = (d\bullet)(X)$.

Consider $\nabla_X Y$ where X and Y are vector fields. We consider $\nabla_X Y$ locally in a chart, i.e. for $\frac{\partial}{\partial x^m}$ a vector field basis:

$$\begin{aligned} \nabla_X Y &= \nabla_{X^i \frac{\partial}{\partial x^i}} \left(Y^m \frac{\partial}{\partial x^m} \right) \\ &= X^i \left[\Big(\nabla_{\frac{\partial}{\partial x^i}} (Y^m) \Big) \frac{\partial}{\partial x^m} + Y^m \nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial}{\partial x^m} \right) \right] \\ &= X^i \Big(\nabla_{\frac{\partial}{\partial x^i}} (Y^m) \Big) \frac{\partial}{\partial x^m} + X^i Y^m \nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial}{\partial x^m} \right). \end{aligned} \tag{130}$$

Note, $\nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial}{\partial x^m} \right)$ will be a vector field, in a basis $\frac{\partial}{\partial x^q}$, as $\frac{\partial}{\partial x^m}$ is a vector field. The vector field $\nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial}{\partial x^m} \right)$ is expanded as a linear combination of coefficient functions and basis vector fields in the chart, that is, $\nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial}{\partial x^m} \right) = \Gamma^q_{(x)mi} \frac{\partial}{\partial x^q}$. The connection coefficient functions on M are Γ^q_{mi} with respect to the chart $(\mathcal{U}, x)$. Observe, we write subscript (x) to denote the explicit dependence of the coefficient functions on the chart $(\mathcal{U}, x)$. Thus, $\nabla_X Y = X^i \Big(\nabla_{\frac{\partial}{\partial x^i}} (Y^m) \Big) \frac{\partial}{\partial x^m} + X^i Y^m \Gamma^q_{mi} \frac{\partial}{\partial x^q}$ as in the case of $X = X^i_{(x)} \left(\frac{\partial}{\partial x^i} \right)_p \in T_pM$. Suppose T is a (p,q)-tensor field and S is a (r,s)-tensor field, then the *tensor field product* is defined as

$$\begin{aligned} (T \otimes S)(\omega^1, \ldots, \omega^{p+r}, X^1, \ldots, X^{q+s}) =& T(\omega^1, \ldots, \omega^p, X^1, \ldots, X^q) \cdot_{C^\infty(M)} \\ & S(\omega^{p+1}, \ldots, \omega^{p+r}, X^{q+1}, \ldots, X^{1+s}). \end{aligned} \tag{131}$$

Similarly, $\nabla_X(T \otimes S) = (\nabla_X T) \otimes S + T \otimes (\nabla_X S)$.

Definition A.6. *Let* $(M, \mathcal{O}, \mathcal{A}, \nabla)$ *be a manifold with connection and* $(\mathcal{U}, x) \in \mathcal{A}$ *a chart from the atlas. Then the connection coefficient functions with respect to* $(\mathcal{U}, x)$ *are the* $(\dim M)^3$*-many functions*

$$\begin{aligned} \Gamma^i_{(x)jk} :& \mathcal{U} \to \mathbb{R} \\ & p \mapsto dx^i \left(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j} \right)(p). \end{aligned} \tag{132}$$

It follows that $\nabla_X Y$ is a vector field for $X, Y \in \Gamma(TM)$ that can be expanded in terms of $\left(\frac{\partial}{\partial x^i} \right)$ with components $(\nabla_X Y)^i$ given as $(\nabla_X Y)^i = X^m \left(\frac{\partial}{\partial x^m} Y^i \right) + \Gamma^i_{nm} Y^n X^m$ for $\nabla_X Y = X^i \left(\frac{\partial}{\partial x^i} Y^m \right) \frac{\partial}{\partial x^m} + X^i Y^m \Gamma^q_{mi} \frac{\partial}{\partial x^q}$.

We summarize these formulae as follows.

(i) For X, Y vector fields, the vector field $\nabla_X Y$ has components given by $(\nabla_X Y)^i = X(Y^i) + \Gamma^i_{jm} Y^j X^m$

(ii) For X a vector field and ω a covector field, $(\nabla_X \omega)_i = X(\omega_i) - \Gamma^j_{im}\omega_j X^m$.

(iii) Similarly, by further application of the Leibnitz property, for a $(1,2)$-tensor field T with components T^i_{jk}, the covariant derivative is $(\nabla_X T)^i_{jk} = X(T^i_{jk}) + \Gamma^i_{sm}T^s_{jk}X^m - \Gamma^s_{jm}T^i_{sk}X^m - \Gamma^s_{km}T^i_{js}X^m$.

A Euclidean space $M = (\mathbb{R}^d, \mathcal{O}_{\text{std.}}, \mathcal{A})$ is a smooth manifold. Suppose $(\mathbb{R}^d, \text{id}_{\mathbb{R}^d}) \subset \mathcal{A}$, then $\Gamma^i_{(x)jk} = dx^i \left(\nabla_{(\text{Euclidean})\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j}\right) = 0$ by the definition of Euclidean space. We could equip $\mathbb{R}^d$ with a hyperbolic connection $\nabla_{\text{hyp.}}$ and, thus, give it curvature such that $\Gamma^i_{(x)jk} = dx^i \left(\nabla_{(\text{Euclidean})\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j}\right) \neq 0$, even though it is still $\mathbb{R}^d$.

Definition A.7. *A vector field X on M is said to be parallely transported along a smooth curve $\gamma : \mathbb{R} \to M$ if $\nabla_{\nu_\gamma} X = 0$ for $\nu_\gamma \in T_pM$.*

The notion of parallelism depends on the covariant derivative ∇.

Definition A.8. *A vector field X on M is said to be parallely transported along a smooth curve $\gamma : \mathbb{R} \to M$ if $\left(\nabla_{\nu_{\gamma,\gamma(\lambda)}} X\right)_{\gamma(\lambda)} = 0$. That is, at each point, the change in the vector field along $\nu_{\gamma,\gamma(\lambda)}$ is zero. Likewise, a vector field is said to be parallel along a curve if $\nabla_{\nu_\gamma} X = \mu X$ for $\mu : \mathbb{R} \to \mathbb{R}$, i.e. $\left(\nabla_{\nu_{\gamma,\gamma(\lambda)}}\right)_{\gamma(\lambda)} = \mu(\lambda) X_{\gamma(\lambda)}$.*

Definition A.9. *Absolute space at time τ is the set of all points of the four-dimensional manifold*

$$\begin{aligned} S_\tau : &= \{p \in M | t(p) = \tau\} \\ &= \{p \in M | t(p) = \tau\}, \end{aligned} \tag{133}$$

which implies that $M = \bigsqcup_\tau S_\tau$. Thus, the spacetime M foliates into S_τ.

Absolute time flows uniformly so $\nabla dt = 0$ on the space of $(0,2)$-tensor fields, recalling that the gradient of a continuously differentiable function $f \in C^\infty(M)$ is $(df)_p : T_pM \xrightarrow{\sim} \mathbb{R}$ and $f \in \Gamma(T^*M)$ is a section. The components of this gradient ∇dt are $\left(\nabla_{\frac{\partial}{\partial x^\mu}} dt\right)_\nu$.

Definition A.10. *A metric g on a smooth manifold $(M, \mathcal{O}, \mathcal{A})$ is a $(0,2)$-tensor field satisfying*

(i) (Symmetry.) $g(X,Y) = g(Y,X)$ for all X, Y vector fields.

(ii) (Non-degeneracy.) The flat map $\flat$ is a C^∞-isomorphism

$$\begin{aligned} \flat : &\Gamma(TM) \to \Gamma(T^*M) \\ &X \mapsto \flat(X) \end{aligned}$$

where $\flat(X)(Y) := g(X,Y)$. If $\flat$ is a C^∞-isomorphism this means it is invertible, whereby $\flat(X) = g(X, \bullet)$.

Definition A.11. *The signature of a metric tensor g is the number of non-negative eigenvalues of the diagonalized form of the tensor with respect to the basis. In particular, if g is a $(0,2)$-tensor* *then it has signature (p,q) if it can be diagonalized as:*

$$\begin{bmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & -1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & -1 & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{bmatrix} \tag{134}$$

where there are p-many $+1$'s and q-many -1's.

The condition that $\flat : \Gamma(TM) \to \Gamma(T^*M)$ is an isomorphism means that there are no zeros in the signature.

Definition A.12. *A metric is called Riemannian if its signature is $(++\cdots+)$. Likewise, a metric is called Lorentzian if it has signature $(+-\cdots-)$.*

Definition A.13. *On a Riemmannian metric manifold $(M, \mathcal{O}, \mathcal{A}, g)$ for g a Riemannian metric, the speed of a curve at $\gamma(\lambda)$ is given by*

$$S(\lambda) := \left(\sqrt{g(\nu_\gamma, \nu_\gamma)}\right)_{\gamma(\lambda)}. \tag{135}$$

Let (M,g) be a metric manifold and let $\gamma : (0,1) \to M$ be a smooth curve. Then the length of γ is given by the scalar

$$L[\gamma] = \int_0^1 d\lambda S[\lambda] = \int_0^1 d\lambda \sqrt{(g(\nu_\gamma, \nu_\gamma))_{\gamma(\lambda)}}. \tag{136}$$

Definition A.14. *A curve $\gamma : (0,1) \to M$ is said to be a geodesic on a Riemannian manifold $(M, \mathcal{O}, \mathcal{A}, g)$ if it is a stationary curve with respect to the length functional L.*

Minkowski space $\mathbb{R}^{n-1,1}$ with signature $(n-1,1)$ is the model for a Lorentzian manifold. In a similar vein, $\mathbb{R}^{p,q}$ is the model space for a pseudo-Riemannian manifold of signature (p,q) with line element

$$ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2. \tag{137}$$

From special relativity, the spacetime interval ds^2 is given by the infinitesimal separation of events [7]:

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = \sum_{\mu,\nu} \eta_{\mu\nu} dx^\mu dx^\nu. \tag{138}$$

Moreover, by the Einstein summation convention, we simply write $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ where $\eta_{\mu\nu}$ is a square matrix with inverse given by

$$(\eta_{\mu\nu})^{-1} = \eta^{\mu\nu} = \begin{bmatrix} -c^{-2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{139}$$

Definition A.15. *A vector field X is said to be a Killing field if the Lie derivative of the metric g with respect to X is zero, i.e. $\mathcal{L}_X g = 0$. In a coordinate chart, this is equivalent to the Killing equation* [3]*:*

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0. \tag{140}$$

Remark A.16. *The connection coefficients Γ^q_{ij} are the coefficient functions of the so-called Levi-Civita connection ${}^{L.C.}\nabla$. This is the connection that identifies geodesics with autoparallels such that $\ddot{\gamma}^q + (g^{-1})^{qm}\frac{1}{2}\left(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij}\right)\dot{\gamma}^i\dot{\gamma}^j = 0$. We make this choice of connection ∇ if g is given, which means from a smooth manifold $(M, \mathcal{O}, \mathcal{A}, g)$, we construct a smooth manifold with metric connection $(M, \mathcal{O}, \mathcal{A}, g, {}^{L.C.}\nabla)$.*

Definition A.17 (Metric compatability)**.** *If $\nabla g = 0$ and torsion vanishes, $T = 0$, for a connection ∇ and metric g, then it follows that $\nabla :={}^{L.C.} \nabla$. That is, we say ∇ is compatible with the metric.*

Definition A.18. *The Riemann-Christoffel curvature is defined by the $(0,4)$-tensor*

$$R_{abcd} := g_{am} R^m_{bcd}. \tag{141}$$

The contraction is done with respect to a chart-independent trace.

Definition A.19. *The Ricci tensor is a contraction of the Riemann curvature tensor $R_{ab} = R^m_{amb}$.*

Definition A.20. *On a metric manifold, the Ricci scalar curvature is the metric-dependent contraction $(g^{-1})^{ab} R_{ab}$.*

Definition A.21. *The torsion of a connection ∇ is the $(1,2)$-tensor field $T(\omega, X, Y) = \omega(\nabla_X Y - \nabla_Y X - [X,Y])$.*

The commutator of vector fields is defined as the vector field $[X,Y] = X(Y) - Y(X)$, which acts on a $C^\infty(M)$-function as $[X,Y] = X(Yf) - Y(Xf)$. Intuitively, Lie algebras $\mathfrak{g}$ are tangent spaces to Lie groups G, i.e. the vector space underlying a Lie algebra is the tangent space to the Lie group. We must check that $T(\omega, X, Y)$ is a tensor field.

Proof. For T to be a tensor field, it must be C^∞-linear in each entry.

(i) (C^∞-scaling.) $T(f\omega, X, Y) = (f\omega)(\nabla_X Y - \nabla_Y X - [X,Y]) = fT(\omega, X, Y)$.

(ii) (Additivity.) $T(\omega + \Psi, X, Y) = (\omega + \Psi)(\nabla_X Y - \nabla_Y X - [X,Y]) = \omega(\nabla_X Y - \nabla_Y X - [X,Y]) + \Psi(\nabla_X Y - \nabla_Y X - [X,Y]) = T(\omega, X, Y) + T(\Psi, X, Y)$.

(iii) (C^∞-scaling.) $T(\omega, fX, Y) = \omega(f\nabla_X Y - (Yf)X - f\nabla_Y X - [fX, Y]) = \omega(f\nabla_X Y - f\nabla_Y X - f[X,Y]) = f\omega(\nabla_X Y - \nabla_Y X - [X,Y]) = fT(\omega, X, Y)$. Similarly, since the commutator $[X,Y]$ is anti-symmetric, $T(\omega, Y, X) = \omega(\nabla_Y X - \nabla_X Y - [Y,X]) = \omega(\nabla_Y X - \nabla_X Y + [X,Y]) = -T(\omega, X, Y)$. Therefore, we have proven that $T(\omega, X, Y)$ is a $(1,2)$-tensor field.

□

Definition A.22. *A manifold $(M, \mathcal{O}, \mathcal{A})$ is called torsion-free if the torsion tensor field vanishes everywhere, i.e. $T = 0$ where $T(\omega, X, Y) = \omega(\nabla_Y X - \nabla_X Y - [X,Y])$.*

Definition A.23. *The Riemann curvature of a connection ∇ is the $(1,3)$-tensor field $Riem(\omega, Z, X, Y) := \omega\left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\right)$. The components of the Riemann curvature are*

$$Riem\left(dx^i, \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}, \frac{\partial}{\partial x^c}\right) = R^i_{abc}.$$

If we apply the directional derivative ∇_Y in direction Y followed by the directional derivative ∇_X in direction X to a vector field Z, then we obtain $\nabla_X \nabla_Y Z$. The difference in applying the directional derivatives in different orders $\bullet(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z) = \text{Riem}(\bullet, Z, X, Y) + \nabla_{[X,Y]} Z$ measures how much $\nabla_X \nabla_Y Z$ fails to commute. In one chart $(\mathcal{U}, x)$, we have $\nabla_{\frac{\partial}{\partial x^a}} \nabla_{\frac{\partial}{\partial x^b}} Z - \nabla_{\frac{\partial}{\partial x^b}} \nabla_{\frac{\partial}{\partial x^a}} Z = \nabla_a \nabla_b Z - \nabla_b \nabla_a Z$. The vector field components are $(\nabla_a \nabla_b Z)^m - (\nabla_b \nabla_a Z)^m = \text{Riem}^m_{nab} Z^n + \nabla_{\left[\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right]} Z$. Note that $\left[\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right] = 0$ by the definition of $\frac{\partial f}{\partial x^i}$, such that

$$(\nabla_a \nabla_b Z)^m - (\nabla_b \nabla_a Z)^m = \text{Riem}^m_{nab} Z^n. \tag{142}$$

In flat space where the Riemann curvature tensor is zero, we can swap the order of the covariant derivatives.

Appendix B Autoparallel Equation Derivation

Definition B.1. *A curve $\gamma : \mathbb{R} \to M$ is called autoparallely transported if $\nabla_{\nu_\gamma} \nu_\gamma = 0$. Likewise, a curve is said to be autoparallel if $\nabla_{\nu_\gamma} \nu_\gamma = 0 = \mu \nu_\gamma$ for $\mu : \mathbb{R} \to \mathbb{R}$.*

Consider an autoparallely transported curve γ and consider the portion of the curve lying in $\mathcal{U}$, where $(\mathcal{U}, x) \in \mathcal{A}$. We would like to express $\nabla_{\nu_\gamma} \nu_\gamma$ in terms of chart representatives. Since $\nu_\gamma \in \Gamma(TM)$, the vector field at a point $\gamma(\lambda)$ along the curve is $\nu_{\gamma,\gamma(\lambda)} = \dot{\gamma}^m_{(x)} \left(\frac{\partial}{\partial x^m}\right)_{\gamma(\lambda)}$ where $\gamma^m_{(x)} = x^m \circ \gamma$ so that

$$\nabla_{\nu_\gamma} \nu_\gamma = \nabla_{\dot{\gamma}^m_{(x)} \left(\frac{\partial}{\partial x^m}\right)_{\gamma(\lambda)}} \dot{\gamma}^n_{(x)} \left(\frac{\partial}{\partial x^n}\right)_{\gamma(\lambda)}. \tag{143}$$

By C^∞-linearity of ∇,

$$\begin{aligned}
\nabla_{\nu_\gamma} \nu_\gamma &= \nabla_{\dot{\gamma}^m_{(x)} \left(\frac{\partial}{\partial x^m}\right)_{\gamma(\lambda)}} \dot{\gamma}^n_{(x)} \left(\frac{\partial}{\partial x^n}\right)_{\gamma(\lambda)} \\
\dot{\gamma}^m_{(x)} &\left(\nabla_{\left(\frac{\partial}{\partial x^m}\right)_{\gamma(\lambda)}} \dot{\gamma}^n_{(x)} \frac{\partial}{\partial x^n}\right) \\
&= \dot{\gamma}^m_{(x)} \left[\left(\nabla_{\left(\frac{\partial}{\partial x^m}\right)_{\gamma(\lambda)}} \dot{\gamma}^n_{(x)}\right) \frac{\partial}{\partial x^n} + \dot{\gamma}^n_{(x)} \left(\nabla_{\left(\frac{\partial}{\partial x^m}\right)_{\gamma(\lambda)}}\right)\right]
\end{aligned} \tag{144}$$

where $\nabla_{\left(\frac{\partial}{\partial x^m}\right)_{\gamma(\lambda)}} \dot{\gamma}(x)^n = \left(\frac{\partial \dot{\gamma}^n_{(x)}}{\partial x^m}\right)_{\gamma(\lambda)}$ for $\nabla_X f = Xf$. Dropping explicit reference to the chart $(\mathcal{U}, x)$,

$$\begin{aligned}
\nabla_{\nu_\gamma} \nu_\gamma &= \dot{\gamma}^m \frac{\partial \dot{\gamma}^n}{\partial x^m} \frac{\partial}{\partial x^n} + \dot{\gamma}^m \dot{\gamma}^n \left(\nabla_{\frac{\partial}{\partial x^m}} \frac{\partial}{\partial x^n}\right) \\
&= \dot{\gamma}^m \frac{\partial \dot{\gamma}^n}{\partial x^m} \frac{\partial}{\partial x^n} + \dot{\gamma}^m \dot{\gamma}^n \Gamma^q_{nm} \frac{\partial}{\partial x^q}.
\end{aligned}$$

Therefore the condition that $\nabla_{\nu_\gamma} \nu_\gamma = 0$ in autoparallel transport becomes

$$\nabla_{\nu_\gamma} \nu_\gamma = \dot{\gamma}^m \frac{\partial \dot{\gamma}^q}{\partial x^m} \frac{\partial}{\partial x^q} + \dot{\gamma}^m \dot{\gamma}^n \Gamma^q_{nm} \frac{\partial}{\partial x^q} = 0.$$

Note, $\frac{\partial \dot{\gamma}^q}{\partial x^m}$ is the projection of the tangent vector components to the curve derived in all possible directions in the direction of the curve, so $\dot{\gamma}^m \frac{\partial \dot{\gamma}^q}{\partial x^m}$ is the second derivative $\ddot{\gamma}^m$. That is, $\ddot{\gamma}^m_{(x)} \frac{\partial}{\partial x^q} + \dot{\gamma}^m_{(x)} \dot{\gamma}^n_{(x)} \Gamma^q_{\ nm} \frac{\partial}{\partial x^q} = \left(\ddot{\gamma}^m_{(x)} + \dot{\gamma}^m_{(x)} \dot{\gamma}^n_{(x)} \Gamma^q_{\ nm} \right) \frac{\partial}{\partial x^q} = 0$. More precisely, the chart representative of a curve γ that is an autoparallel satisfies:

$$\ddot{\gamma}^m_{(x)}(\lambda) + \Gamma^m_{(x)ab}(\gamma(\lambda)) \dot{\gamma}^a_{(x)}(\lambda) \dot{\gamma}^b_{(x)}(\lambda) = 0. \tag{145}$$

Alternatively, we consider the trajectory $\gamma : \mathbb{R} \to M$ of a particle in the spacetime M. In a chart $(\mathcal{U}, x)$, the Euler–Lagrange equations take the form:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^m} \right) - \frac{\partial \mathcal{L}}{\partial x^m} = 0. \tag{146}$$

We have a Lagrangian representative in the chart that takes the components of the curves $\gamma^i_{(x)}$ in a chart and the components of the velocity $\dot{\gamma}^i_{(x)} = (x^i \circ \gamma)'$ in T_pM as arguments: $\mathcal{L}(\gamma^i, \dot{\gamma}^i) = \sqrt{g_{ij}(\gamma(\lambda)) \dot{\gamma}^i(\lambda) \dot{\gamma}^j(\lambda)}$. Applying the Euler–Lagrange equations, we obtain

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{\gamma}^m} &= \frac{1}{2\sqrt{g_{ij}(\gamma(\lambda)) \dot{\gamma}^i(\lambda) \dot{\gamma}^j(\lambda)}} 2 g_{mj}(\gamma(\lambda)) \dot{\gamma}^j(\lambda) \\ &= \frac{g_{mj}(\gamma(\lambda)) \dot{\gamma}^j(\lambda)}{\sqrt{g_{ij}(\gamma(\lambda)) \dot{\gamma}^i(\lambda) \dot{\gamma}^j(\lambda)}} \end{aligned} \tag{147}$$

such that

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\gamma}^m} \right) = & \frac{d}{dt} \left(\frac{1}{\sqrt{g_{ij}(\gamma(\lambda)) \dot{\gamma}^i(\lambda) \dot{\gamma}^j(\lambda)}} \right) g_{mj}(\gamma(\lambda)) \dot{\gamma}^j(\lambda) \\ & + \frac{1}{\sqrt{g_{ij}(\gamma(\lambda)) \dot{\gamma}^i(\lambda) \dot{\gamma}^j(\lambda)}} \frac{d}{dt} \left(g_{mj}(\gamma(\lambda)) \dot{\gamma}^j(\lambda) \right). \end{aligned} \tag{148}$$

Since $\gamma(\lambda) = (\gamma^1(\lambda), \dots, \gamma^{\dim M}(\lambda))$, the time derivative becomes:

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\gamma}^m} \right) = & \frac{d}{dt} \left(\frac{1}{\sqrt{g_{ij}(\gamma(\lambda)) \dot{\gamma}^i(\lambda) \dot{\gamma}^j(\lambda)}} \right) g_{mj}(\gamma(\lambda)) \dot{\gamma}^j(\lambda) \\ & + \frac{1}{\sqrt{g_{ij}(\gamma(\lambda)) \dot{\gamma}^i(\lambda) \dot{\gamma}^j(\lambda)}} \left(g_{mj}(\gamma(\lambda)) \ddot{\gamma}^j(\lambda) + (\partial_s g_{mj} \dot{\gamma}^s \dot{\gamma}^j(\lambda)) \right). \end{aligned} \tag{149}$$

We now impose the condition that at each point $g(\dot{\gamma}, \dot{\gamma}) = 1$, which does not change $L[\gamma]$ by the reparameterization theorem. Thus, $\frac{d}{dt} \left(\frac{1}{\sqrt{g_{ij}(\gamma(\lambda)) \dot{\gamma}^i(\lambda) \dot{\gamma}^j(\lambda)}} \right) = 0$ so

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\gamma}^m} \right) = g_{mj}(\gamma(\lambda)) \ddot{\gamma}^j(\lambda) + (\partial_s g_{mj} \dot{\gamma}^s \dot{\gamma}^j(\lambda)). \tag{150}$$

The derivative of the Lagrangian with respect to γ^m is given by

$$\frac{\partial \mathcal{L}}{\partial \gamma^m} = \frac{1}{2\sqrt{g_{ij}(\gamma(\lambda)) \dot{\gamma}^i(\lambda) \dot{\gamma}^j(\lambda)}} \partial_m g_{ij}(\gamma(\lambda)) \dot{\gamma}^i(\lambda) \dot{\gamma}^j(\lambda) \tag{151}$$

where, again under reparameterization $g(\dot{\gamma}, \dot{\gamma}) = 1$, we have

$$\frac{\partial \mathcal{L}}{\partial \gamma^m} = \frac{1}{2} \partial_m g_{ij}(\gamma(\lambda)) \dot{\gamma}^i(\lambda) \dot{\gamma}^j(\lambda). \tag{152}$$

Writing this in terms of the Euler–Lagrange equations of motion $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\gamma}^m}\right) - \frac{\partial \mathcal{L}}{\partial \gamma^m} = 0$:

$$g_{mj} \ddot{\gamma}^j + \partial_s g_{mj} \dot{\gamma}^s \dot{\gamma}^j - \frac{1}{2} \partial_m g_{ij} \dot{\gamma}^i \dot{\gamma}^j = 0. \tag{153}$$

We use the so-called inverse metric $(g^{-1})^{qm}$ and apply it to both sides, summing over m by the Einstein summation convention:

$$\begin{aligned} (g^{-1})^{qm} g_{mj} \ddot{\gamma}^j + (g^{-1})^{qm} \partial_s g_{mj} \dot{\gamma}^s \dot{\gamma}^j - \frac{1}{2} (g^{-1})^{qm} \partial_m g_{ij} \dot{\gamma}^i \dot{\gamma}^j = 0 \\ \delta^q_j \ddot{\gamma}^j + (g^{-1})^{qm} \partial_i g_{mj} \dot{\gamma}^i \dot{\gamma}^j - \frac{1}{2} (g^{-1})^{qm} \partial_m g_{ij} \dot{\gamma}^i \dot{\gamma}^j = 0 \\ \ddot{\gamma}^q + (g^{-1})^{qm} \left(\partial_i g_{mj} - \frac{1}{2} \partial_m g_{ij} \right) \dot{\gamma}^i \dot{\gamma}^j = 0. \end{aligned} \tag{154}$$

Since $\dot{\gamma}^i$ and $\dot{\gamma}^j$ are symmetric, $\partial_i g_{mj} = \partial_j g_{mi}$ so $\frac{1}{2} \partial_i g_{mj} + \frac{1}{2} \partial_j g_{mi} = \partial_i g_{mj}$ such that

$$\ddot{\gamma}^q + (g^{-1})^{qm} \frac{1}{2} \left(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij} \right) \dot{\gamma}^i \dot{\gamma}^j. \tag{155}$$

In fact, $\frac{1}{2}(g^{-1})^{qm} \left(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij} \right)$ is the Christoffel symbol of second-kind $\Gamma^q_{ij}(\gamma(\lambda))$. Thus, the geodesic equation describing the trajectory γ of a particle in a local chart of spacetime is given by

$$\ddot{\gamma}^m_{(x)}(\lambda) + \Gamma^m_{(x)ab}(\gamma(\lambda)) \dot{\gamma}^a \dot{\gamma}^b. \tag{156}$$

Appendix C Laurent Development for Hyperbolic Trigonometric Functions

The Weierstrass product for hyperbolic sine is

$$\frac{\sinh z}{z} = \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{\pi^2 n^2} \right), \tag{157}$$

from which it follows that

$$\log \sinh z - \log z = \sum_{n=1}^{\infty} \log \left(1 + \frac{z^2}{\pi^2 n^2} \right). \tag{158}$$

Differentiating this expression, we obtain the logarithmic derivative of Equation (157):

$$\coth z - \frac{1}{z} = \sum_{n=1}^{\infty} \frac{2z}{\pi^2 n^2 + z^2}. \tag{159}$$

We expand this identity as a geometric series and thereby obtain the Laurent series expansion of hyperbolic cotangent in terms of the Riemann zeta function in a punctured ε-neighborhood of $z=-1$. Thus,

$$\begin{aligned}\coth(z) &= \frac{1}{z} - 2\sum_{n=1}^{\infty}(-1)^n\ \frac{\zeta\left(2n\right)}{\pi^{2n}}z^{2n-1} \\ &= \frac{1}{z} - 2\sum_{n=1}^{\infty}(-1)^n\ \frac{B_{2n}}{\pi^{2n}}z^{2n-1}\end{aligned}$$

for $|z|< 1$, where we recognize $\zeta(2m)$ as the Bernoulli numbers B_{2m}. Up to third order, the Laurent development is simply:

$$\coth(z) = \frac{1}{z} + 2\frac{\zeta(2)}{\pi^2}z - 2\frac{\zeta(4)}{\pi^4}z^3 + O(z^5) = \frac{1}{z} + \frac{z}{3} - \frac{z^3}{45} + O(z^5).$$

Appendix D Euler–Lagrange Equations for Electromagnetic Field Theory

Consider a massive point particle with an action

$$S[\gamma; A] = \int d\lambda\left(m\sqrt{g_{\gamma,\gamma(\lambda)}(\nu_{\gamma,\gamma(\lambda)},\nu_{\gamma,\gamma(\lambda)})} + qA(\nu_{\gamma,\gamma(\lambda)})\right) \tag{160}$$

where A is a covector field, i.e. a $(0,1)$-tensor field, on M. For instance $A:\Gamma(TM)\xrightarrow{\sim}C^\infty(M)$ could be the electromagnetic potential. The term $L_{\text{int}}(\gamma,\dot\gamma,A):=qA(\nu_{\gamma,\gamma(\lambda)})$ is an L-interaction term satisfying $\frac{d}{dt}\left(\frac{\partial L_{\text{int}}}{\partial\dot\gamma^a_{(x)}}\right) - \frac{\partial L_{\text{int}}}{\partial\gamma^a_{(x)}}$ for $\gamma^m_{(x)} = x^m\circ\gamma$ in a chart. If we ignore the coupling term, the Euler–Lagrange equation would be $m\nabla_{\nu_\gamma}\nu_\gamma = 0$. However, since $L_{\text{int}} = qA_{(x)m}\dot\gamma^m_{(x)}$ with the interaction term, the new equations of motion become

$$m\left(\nabla_{\nu_\gamma}\nu_\gamma\right)_a + \frac{d}{dt}\left(\frac{\partial L_{\text{int}}}{\partial\dot\gamma^a_{(x)}}\right) - \frac{\partial L_{\text{int}}}{\partial\gamma^a_{(x)}} = 0. \tag{161}$$

It follows that $\frac{\partial L_{\text{int}}}{\partial\dot\gamma^a} = qA_{(x)a}$. Since the potential $A_{(x)a}$ depends on the position along the worldline $\gamma^m = x^m\circ\gamma$, we have that

$$\frac{d}{dt}\left(\frac{\partial L_{\text{int}}}{\partial\dot\gamma^a}\right) = q\frac{d}{dt}\left(qA_{(x)a}(x^m\circ\gamma)\right) = q\frac{\partial}{\partial x^m}(A_{(x)a})\dot\gamma^m_{(x)} \tag{162}$$

and

$$\frac{d}{dt}\left(\frac{\partial L_{\text{int}}}{\partial\gamma^a}\right) = q\frac{\partial}{\partial x^a}(A_{(x)m})\dot\gamma^m_{(x)}. \tag{163}$$

Then

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L_{\text{int}}}{\partial\dot\gamma^a_{(x)}}\right) - \frac{\partial L_{\text{int}}}{\partial\gamma^a_{(x)}} &= q\frac{\partial}{\partial x^m}(A_{(x)a})\dot\gamma^m_{(x)} - q\frac{\partial}{\partial x^a}(A_{(x)m})\dot\gamma^m_{(x)} \\ &= q\left(\frac{\partial A_a}{\partial\dot\gamma^a_{(x)}} - \frac{\partial A_m}{\partial x^a}\right)\dot\gamma^m_{(x)} = qF_{(x)am}\dot\gamma^m_{(x)}\end{aligned} \tag{164}$$

where F is the Faraday tensor which describes the Lorentz force of a charged particle in an electromagnetic field. Therefore, the equation of motion is:

$$m(\nabla_{\nu_\gamma}\nu_\gamma)^a = -qF^a_m\dot{\gamma}^m \tag{165}$$

where $F^a_{(x)m} = \frac{\partial A^a_{(x)}}{\partial x^m} - \frac{\partial A_{(x)m}}{\partial x^a}$. If we let $\flat : \Gamma(TM) \to \Gamma(T^*M)$ then the chart-independent formulation of $m(\nabla_{\nu_\gamma}\nu_\gamma)^a = -qF^a_m\dot{\gamma}^m$ is $m(\nabla_{\nu_\gamma}\nu_\gamma)^a = -q\flat^{-1}(F(\bullet, \nu_\gamma))$ for F a covector field in $\Gamma(T^*M)$. Classical field matter is any tensor field on spacetime whose equations of motion are derived from the action.

For instance, consider the Maxwell electromagnetic field theory with an action given in terms of the electromagnetic $(0, 1)$-tensor field potential A and a fixed metric g:

$$S_{\text{Maxwell}}[A; g] = \frac{1}{4}\int_M d^4x\sqrt{-g}F_{ab}F_{cd}g^{ac}g^{bd}. \tag{166}$$

We introduce the volume form determinant $\sqrt{-g} = \sqrt{-\det((g_{(x)ij})(x^{-1}(\alpha)))} := \omega_{(x)}$ for $\alpha \in \mathbb{R}^4$ to yield a proper notion of integration on the Lorentzian manifold M, using the signature $(+ - --)$. The Faraday tensor is $F_{ab} = 2\partial_{[a}A_{b]} = 2(\nabla_{[a}A)_{b]}$. Let $\mathcal{L} := \sqrt{-g}F_{ab}F_{cd}g^{ac}g^{bd}$ be the Lagrangian density. The Euler–Lagrange equations for fields give

$$\frac{\partial\mathcal{L}}{\partial A_m} - \frac{\partial}{\partial x^s}\left(\frac{\partial\mathcal{L}}{\partial(\partial_s A_m)}\right) + \frac{\partial}{\partial x^s}\frac{\partial}{\partial x^t}\frac{\partial^2\mathcal{L}}{\partial(\partial_t\partial_s A_m)} - \dots. \tag{167}$$

The higher-order terms vanish because the Lagrangian density only depends on a derivative of A up to first order. Thus,

$$\frac{\partial\mathcal{L}}{\partial A_m} - \frac{\partial}{\partial x^s}\left(\frac{\partial\mathcal{L}}{\partial(\partial_s A_m)}\right) = 0 \tag{168}$$

for $S_{\text{Maxwell}}[A; g]$. The inhomogeneous Maxwell equation without a current is

$$\left(\nabla_{\frac{\partial}{\partial x^m}}F\right)^{ma} = 0. \tag{169}$$

The inhomogeneous field theory in which the field is coupled to a current $j = q\nu_\gamma$ has an associated action

$$S_{\text{Maxwell}}[A; g] = \frac{1}{4}\int_M d^4x\sqrt{-g}(F_{ab}F_{cd}g^{ac}g^{bd} + qA(j)) \tag{170}$$

which, under first-order variation, yields the equations of motion:

$$\left(\nabla_{\frac{\partial}{\partial x^m}}F\right)^{ma} = j^a. \tag{171}$$

Appendix E Equations of Motion for Worldline in the Presence of Gravity

By Newton's second law, the acceleration of a worldline is $\nabla_{\nu_X}\nu_X = \frac{F}{m}$ where F is a spatial vector field, i.e. $dt(F) = 0$. It follows that $a^m = (\nabla_{\nu_X}\nu_X)^m = \ddot{X}^m + \Gamma^m_{bc}\dot{X}^b\dot{X}^c$. Consider Newton's second law in a stratified chart $\mathcal{A}_{\text{stratified}}$. Parameterize the worldline $X : \mathbb{R} \to \mathbb{R}^4$ with respect to λ. Then $\nabla_{\nu_X}\nu_X = \frac{F}{m}$ is equivalent to

$$\begin{aligned} \ddot{X}^0 + \Gamma^0_{cd}\dot{X}^c\dot{X}^d &= 0 \\ \ddot{X}^\alpha + \Gamma^\alpha_{cd}\dot{X}^c\dot{X}^d &= 0. \end{aligned} \tag{172}$$

Assume that the manifold is torsion-free so $T\left(dx^i, \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right) = 2\Gamma^i_{[a,b]} = 0$. Then we decompose $\Gamma^\alpha_{cd}\dot{X}^c\dot{X}^d = \Gamma^\alpha_{\gamma\delta}\dot{X}^\gamma\dot{X}^\delta + \Gamma^\alpha_{00}\dot{X}^0\dot{X}^0 + \Gamma^\alpha_{\gamma 0}\dot{X}^\gamma\dot{X}^0 + \Gamma^\alpha_{0\delta}\dot{X}^0\dot{X}^\delta$ where Greek indices run between $1, 2, 3$. More precisely,

$$\begin{aligned} \ddot{X}^\alpha + \Gamma^\alpha_{cd}\dot{X}^c\dot{X}^d &= 0 \\ \ddot{X}^\alpha + \Gamma^\alpha_{\gamma\delta}\dot{X}^\gamma\dot{X}^\delta + \Gamma^\alpha_{00}\dot{X}^0\dot{X}^0 + 2\Gamma^\alpha_{\gamma 0}\dot{X}^\gamma\dot{X}^0 &= \frac{F^\alpha}{m}. \end{aligned} \tag{173}$$

Observe that $\Gamma^0_{cd} = 0$ in a stratified atlas so Newton's equations become:

$$\begin{aligned} \ddot{X}^0 &= 0 \\ \ddot{X}^\alpha + \Gamma^\alpha_{\gamma\delta}\dot{X}^\gamma\dot{X}^\delta + \Gamma^\alpha_{00}\dot{X}^0\dot{X}^0 + 2\Gamma^\alpha_{\gamma 0}\dot{X}^\gamma\dot{X}^0 &= \frac{F^\alpha}{m}. \end{aligned} \tag{174}$$

Since $X^0(\lambda) = (x^0 \circ X)(\lambda) = (t \circ X)(\lambda) = a\lambda + b$ for $a, b \in \mathbb{R}$ in a stratified atlas, we have that $d/d\lambda = ad/dt$ and $d^2/d\lambda^2 = a^2 d/dt^2$ so $\ddot{X}^\alpha + \Gamma^\alpha_{\gamma\delta}\dot{X}^\gamma\dot{X}^\delta + \Gamma^\alpha_{00}\dot{X}^0\dot{X}^0 + 2\Gamma^\alpha_{\gamma 0}\dot{X}^\gamma\dot{X}^0 = \frac{F^\alpha}{m}$ for $\alpha = 1, 2, 3$ becomes

$$\ddot{X}^\alpha + \Gamma^\alpha_{\gamma\delta}\dot{X}^\gamma\dot{X}^\delta + \Gamma^\alpha_{00}\dot{X}^0\dot{X}^0 + 2\Gamma^\alpha_{\gamma 0}\dot{X}^\gamma\dot{X}^0 = \frac{F^\alpha}{ma^2}. \tag{175}$$

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